Module Tests

Module 2 Test 1:

1 (a) sin 2x + sin 3x + sin 4x = 0.

sin 4x + sin 2x + sin 3x = 0.

$\Rightarrow 2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)+\sin 3x = 0$

$\Rightarrow$ 2 sin 3x cos x + sin 3x = 0

$\Rightarrow$ sin 3x [2 cos x + 1] = 0

$\sin 3x = 0, \quad \cos x = -\frac{1}{2}$

$3x = n\pi \quad x = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$

$x = \frac{n\pi}{3}, n \in \mathbb{Z}$

$\left.\begin{array}{l}\therefore x = \frac{n\pi}{3} \\ \quad x = 2\left(n\pi \pm \frac{\pi}{3}\right)\end{array}\right\} n \in \mathbb{Z}$

(b) sin A + sin B + sin C

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin(180-(A+B))$ since C = 180 – (A + B)

$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin(A+B)$

$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

$= 2\sin\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right]$

$= 2\sin\left(\frac{A+B}{2}\right)\left[2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right]$

$= 4\sin\left(90-\frac{C}{2}\right)\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)$ since $\frac{A+B}{2} = \left(90-\frac{C}{2}\right)$

$= 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$ since $\sin\left(90-\frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$

(c) (i) 2sin θ + cos θ = R sin (θ + α)

= R sin θ cos α + R cos θ sin α

$\left.\begin{array}{l}R\cos\alpha = 2 \\ R\sin\alpha = 1\end{array}\right\} \Rightarrow \tan\alpha = \frac{1}{2}, \alpha = 26.6°$

$R^2 = 2^2 + 1^2 \Rightarrow R = \sqrt{5}$

$\therefore 2\sin\theta + \cos\theta = \sqrt{5}\ \sin(\theta + 26.6°)$

(ii) 2 sin θ + cos θ = 2

$\Rightarrow \sqrt{5}\sin(\theta + 26.6) = 2$

$$\theta + 26.6 = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\theta + 26.6° = 180n + (-1)^n (63.4),\ n \in \mathbb{Z}$$

$$\left.\begin{aligned}\theta &= 360°n + 36.8° \\ \theta &= 180°(2n+1) - 90°\end{aligned}\right\} n \in \mathbb{Z}$$

(iii) $\text{Max} = \dfrac{1}{4 - \sqrt{5}}$

2 (a) P(2, 3), Q(4, –1) R(3, –1)

Equation: $(x - a)^2 + (y - b)^2 = r^2$

$(2 - a)^2 + (3 - b)^2 = r^2$ [1]

$(4 - a)^2 + (-1 - b)^2 = r^2$ [2]

$(3 - a)^2 + (-1 - b)^2 = r^2$ [3]

$[2] - [3] \Rightarrow (4 - a)^2 - (3 - a)^2 = 0$

$16 - 8a + \cancel{a^2} - 9 + 6a - \cancel{a^2} = 0$

$7 - 2a = 0$

$a = \dfrac{7}{2}$

Substitute $a = \dfrac{7}{2}$, $\left(2 - \dfrac{7}{2}\right)^2 + (3 - b)^2 = \left(4 - \dfrac{7}{2}\right)^2 + (-1 - b)^2$

$\dfrac{9}{4} + 9 - 6b + \cancel{b^2} = \dfrac{1}{4} + 1 + 2b + \cancel{b^2}$

$10 = 8b$

$b = \dfrac{10}{8} = \dfrac{5}{4}$

centre $\left(\dfrac{7}{2}, \dfrac{5}{4}\right)$

$r^2 = \dfrac{9}{4} + \left(3 - \dfrac{5}{4}\right)^2 = \dfrac{85}{16} \Rightarrow r = \dfrac{\sqrt{85}}{4}$

Equation is $\left(x - \dfrac{7}{2}\right)^2 + \left(y - \dfrac{5}{4}\right)^2 = \dfrac{65}{16}$

(b) $x = 1 + 4\cos\theta$

$\cos\theta = \dfrac{x - 1}{4}$

$y = -2 + 4\sin\theta$

$\sin\theta = \dfrac{y + 2}{4}$

$\sin^2\theta + \cos^2\theta = \dfrac{(y + 2)^2}{16} + \dfrac{(x - 1)^2}{16}$

$\Rightarrow \dfrac{(x - 1)^2}{16} + \dfrac{(y + 2)^2}{16} = 1$

$\Rightarrow (x - 1)^2 + (y + 2)^2 = 16 = 4^2$

Circle centre (1, –2) radius 4

(c) $x = 2 + \cos t$, $y = 3 + 2\sin t$

$\cos t = x - 2$

$\sin t = \frac{y-3}{2}$

$\sin^2 t + \cos^2 t = \frac{(y-3)^2}{2^2} + (x-2)^2$

$\Rightarrow 1 = (x-2)^2 + \frac{(y-3)^2}{2^2}$

Which is an ellipse with centre (2, 3)

3 (a) (i) $\cos\theta = \frac{\begin{pmatrix}1\\2\\1\end{pmatrix}.\begin{pmatrix}-2\\1\\1\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\left|\begin{pmatrix}-2\\1\\1\end{pmatrix}\right|}$

$= \frac{1}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$

$\Rightarrow \theta = 80.4°$

$\therefore$ angle AOB = 80.4°

(ii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-2\\1\\1\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-3\\-1\\0\end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}p\\2\\5\end{pmatrix} - \begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}p+2\\1\\4\end{pmatrix}$

Since $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$

$\Rightarrow \overrightarrow{AB}.\ \overrightarrow{BC} = 0$

$\begin{pmatrix}-3\\-1\\0\end{pmatrix}.\begin{pmatrix}p+2\\1\\4\end{pmatrix} = 0$

$\Rightarrow -3p - 6 - 1 = 0$

$3p = -7$

$p = \frac{-7}{3}$

(b) (i) $x^2 + y^2 - 8y - 9 = 0$

$y = 11 - x$

$x^2 + (11-x)^2 - 8(11-x) - 9 = 0$

$x^2 + 121 - 22x + x^2 - 88 + 8x - 9 = 0$

$2x^2 - 14x + 24 = 0$

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x = 3, 4$

when $x = 3$, $y = 8$

$x = 4$, $y = 7$

A = (3, 8), B = (4, 7)

Mid point of $AB = \left(\frac{7}{2}, \frac{15}{2}\right)$

(ii) Gradient of $AB = \frac{7-8}{4-3} = -1$

Gradient of the $\perp = 1$

Equation of $\perp$ bisector is:

$$y - \frac{15}{2} = x - \frac{7}{2}$$

$$y = x - \frac{7}{2} + \frac{15}{2}$$

$y = x + 4$

(c) $25x^2 + 9y^2 = 225$

$$\div 225 \Rightarrow \frac{25x^2}{225} + \frac{9y^2}{225} = 1$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$$

Centre (0, 0)

Length of the major axis is 10 units

Parametric equation:

$x = 3 \cos \theta$, $y = 5 \sin \theta$

4 (a)

$$r = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2t \\ 2+t \\ 4+t \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} x &= 1-2t \\ y &= 2+t \\ z &= 4+t \end{aligned}\right\} t \in \mathbb{R} \quad \text{parametric equations}$$

(b)

$$r \,.\, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 8$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 8 \Rightarrow 2x + 3y + 2z = 8$$ is the Cartesian equation of the plane

(c) Substitute $x = 1 - 2t$, $y = 2 + t$, $z = 4 + t$ into the equation of the plane:

$2(1 - 2t) + 3(2 + t) + 2(4 + t) = 8$

$2 - 4t + 6 + 3t + \cancel{8} + 2t = \cancel{8}$

$t = -8$.

(d) Substituting $t = -8$ into the line we get

$x = 1 + 16 = 17$

$y = 2 - 8 = -6$

$z = 4 - 8 = -4$

intersection is $\begin{pmatrix} 17 \\ -6 \\ -4 \end{pmatrix}$

5 (a) $\cos 5x + \cos x + \cos 3x + \cos 7x = 0$

$\Rightarrow 2 \cos 3x \cos 2x + 2 \cos 5x \cos 2x = 0$

$2 \cos 2x (\cos 3x + \cos 5x) = 0$

$4 \cos 2x \cos 4x \cos x = 0$

$\cos x = 0, \ \cos 2x = 0, \ \cos 4x = 0$

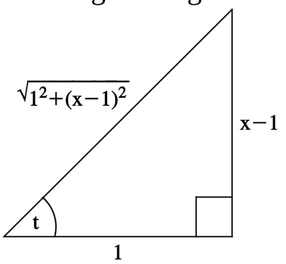
$x = 2n\pi \pm \frac{\pi}{2} \quad 2x = 2n\pi \pm \frac{\pi}{2} \quad 4x = 2n\pi \pm \frac{\pi}{2}$

$$\left.\begin{aligned} x &= 2n\pi \pm \frac{\pi}{2} \\ x &= n\pi \pm \frac{\pi}{4} \\ x &= \frac{1}{2}n\pi \pm \frac{\pi}{8} \end{aligned}\right\} n \in \mathbb{Z}$$

(b) $x = 1 + \tan t$

$\tan t = x - 1$

Drawing a triangle:



$$\cos t = \frac{1}{\sqrt{1+(x-1)^2}}.$$

$y = 2 + \cos t.$

$$y = 2 + \frac{1}{\sqrt{1+(x-2)^2}}$$

Module 2 Test 2

1 (a) (i) $x^2 - 4x + y^2 - 9y - 12 = 0$

$(x-2)^2 - 4 + (y-3)^2 - 9 - 12 = 0$

$(x-2)^2 + (y-3)^2 = 25$

Circle centre (2, 3) radius 5

(ii) $\text{Gradient} = -\left(\frac{2-6}{3-0}\right) = \frac{4}{3}$

$\text{Equation: } y - 0 = \frac{4}{3}(x-6)$

$3y = 4x - 24$

$3y - 4x + 24 = 0$

(iii) $y = x + 8$

$(x-2)^2+(x+8-3)^2=25$

$x^2-4x+4+x^2-10x+25=25$

$2x^2-14x+4=0$

$x^2-7x+2=0$

$x=\dfrac{7\pm\sqrt{41}}{2}$

(b) $x = 1 - t$

$y = 2 + 4t$

$z = 2 + t$

$2 + 3s = 1 - t$ [1]

$1 - s = 2 + 4t$ [2]

$s = 2 + t$ [3]

[1] + [3] $\Rightarrow 2 + 4s = 3$

$s=\dfrac{1}{4}$

Substitute into [3] $\Rightarrow$ $t=\dfrac{1}{4}-2=-\dfrac{7}{4}$

Substitute $s=\dfrac{1}{4},\ t=-\dfrac{7}{4}$ into [2]

$\Rightarrow 1-\dfrac{1}{4}=2-\dfrac{28}{4}$

$\dfrac{3}{4}=\dfrac{-20}{4}$ inconsistent

Since the lines are not parallel and do not intersect, they are skew lines

2 (a) $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$

$\tan 2(22.5)=\dfrac{2\tan 22.5}{1-\tan^2 22.5}$

$\Rightarrow 1=\dfrac{2\tan 22.5}{1-\tan^2 22.5}$

$\Rightarrow 1-\tan^2(22.5)=2\tan(22.5)$

$\tan^2(22.5)+2\tan(22.5)-1=0$

$\tan(22.5)=\dfrac{-2\pm\sqrt{8}}{2}$

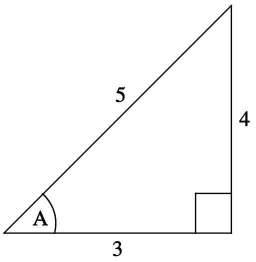
$=\dfrac{-2\pm 2\sqrt{2}}{2}$

$=-1\pm\sqrt{2}$

Since 22.5° is in the first quadrant

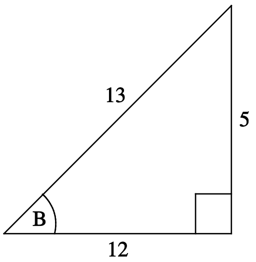
$\tan(22.5°)=-1+\sqrt{2}$

(b)



$\cos A = \frac{3}{5}$

$\tan A = \frac{4}{3}$



$\sin B = \frac{5}{13}$

$\tan B = \frac{5}{12}$

(i) sin (A +B) = sin A cos B + cos A sin B

$= \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$

$= \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$

(ii) cos(A + B) = cos A cos B + sin A sin B

$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) = \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$

(c) $\frac{\sin\theta + \sin 3\theta - \sin 2\theta}{\cos\theta + \cos 3\theta - \cos 2\theta}$

$= \frac{2\sin 2\theta\cos\theta - \sin 2\theta}{2\cos 2\theta\cos\theta - \cos 2\theta}$

$= \frac{\sin 2\theta\ [2\cos - 1]}{\cos 2\theta\ [2\cos - 1]}$

$= \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

3 (a) x = 4 + 2 tan t

$\tan t = \frac{x-4}{2}$

$\tan^2 t = \frac{(x-4)^2}{2^2}$ [1]

y = 3 + sec t

sec t = y – 3

$\sec^2 t = (y-3)^2$ [2]

$$[2]-[1]\Rightarrow \sec^2 t-\tan^2 t=(y-3)^2-\frac{(x-4)^2}{2^2}$$

$$1=(y-3)^2-\frac{(x-4)^2}{4}$$

$$\therefore \text{Cartesian equation is } (y-3)^2-\frac{(x-4)^2}{4}=1$$

(b) $x + 2y + z = 4$

$-x + y + 2z = 6$

$$\cos\theta=\frac{\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\left|\begin{pmatrix}-1\\1\\2\end{pmatrix}\right|}=\frac{3}{\sqrt{6}\sqrt{6}}=\frac{3}{6}=\frac{1}{2}$$

$\theta = 60°$

4 (a) (i) $y^2 = 16x$

$(t^2, 4t) \Rightarrow$ Gradient of the tangent is $\frac{2}{t}$

Equation of the tangent:

$$y-4t=\frac{2}{t}(x-t^2)$$

$ty - 4t^2 = 2x - 2t^2$

$ty - 2x = 4t^2 - 2t^2$

$ty - 2x = 2t^2$

(ii) At $t = 3$, $3y - 2x = 18$ [1]

$t=\frac{1}{3}, \frac{1}{3}y-2x=\frac{2}{9}$ [2]

$$[1]-[2]\Rightarrow \frac{8}{3}y=\frac{160}{9}$$

$$y=\frac{20}{3},\ x=\frac{1}{2}$$

(b) (i) $\cos 5\theta + \cos\theta + 2\cos 3\theta$

$= 2\cos 3\theta \cos 2\theta + 2\cos 3\theta$

$= 2\cos 3\theta(\cos 2\theta + 1)$

$= 2\cos 3\theta(2\cos^2\theta)$

$= 4\cos^2\theta \cos 3\theta$

(ii) $4\cos^2\theta\cos 3\theta = 0$

$\Rightarrow \cos^2\theta = 0, \cos 3\theta = 0$

$\cos\theta = 0 \quad \cos 3\theta = 0$

$$\theta=2n\pi\pm\frac{\pi}{2},\quad 3\theta=2n\pi\pm\frac{\pi}{2}$$

$$\theta=\frac{2n\pi}{3}\pm\frac{\pi}{6}$$

$$\left.\begin{aligned}\therefore \theta &= 2n\pi \pm \frac{\pi}{2} \\ &\frac{2n\pi}{3} \pm \frac{\pi}{6}\end{aligned}\right\} n \in \mathbb{Z}$$

(c) $x^2 + y^2 - 7x + 2y + a = 0$

$x = 7, y = 1$

$\Rightarrow \cancel{49} + 1 - \cancel{49} + 2 + a = 0$

$a = -3$

$x^2 - 7x + y^2 + 2y - 3 = 0$

$$\left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + (y+1)^2 - 1 - 3 = 0$$

$$\left(x - \frac{7}{2}\right)^2 + (y+1)^2 = \frac{49}{4} + 4 = \frac{65}{4}$$

$$\text{Centre}\left(\frac{7}{2}, -1\right)$$

$$\text{Gradient} = \frac{1-(-1)}{7 - \frac{7}{2}} = \frac{2}{\frac{7}{2}} = \frac{4}{7}$$

$$\text{Equation: } y - 1 = \frac{4}{7}(x - 7)$$

$7y - 7 = 4x - 28$

$7y = 4x - 21$

Let the coordinates be (x_1, y_1)

$$\frac{x_1 + 7}{2} = \frac{7}{2} \Rightarrow x_1 = 0$$

$$\frac{y_1 + 1}{2} = -1 \Rightarrow y_1 = -3$$

$(0, -3)$

Module 3 Test 1

1 (a) (i) $$\frac{d}{dx}\left[(x+1)(4x^2+1)^{1/2}\right]$$

$$= (x+1)\frac{1}{2}(8x)(4x^2+1)^{-1/2} + (4x^2+1)^{1/2}$$

$$= \frac{4x(x+1)}{\sqrt{4x^2+1}} + \sqrt{4x^2+1}$$

$$= \frac{4x(x+1) + 4x^2 + 1}{\sqrt{4x^2+1}}$$

$$= \frac{8x^2 + 4x + 1}{\sqrt{4x^2+1}}$$

(ii) $$\frac{d}{dx}\left[\cos^3(3x-2)\right] = 3\cos^2(3x-2)\left[3(-\sin(3x-2))\right]$$

$$= -9\cos^2(3x-2)\sin(3x-2)$$

(b) (i) $\int_0^2 f(x)\,dx = 8$

$\int_0^2 [x^2 - f(x)]\,dx = \left[\frac{1}{3}x^3\right]_0^2 - \int_0^2 f(x)dx$

$= \frac{8}{3} - 8 = \frac{-16}{3}$

(ii) $y = x^2 + 2$
$y = 14 - x$
$x^2 + 2 = 14 - x$
$x^2 + x - 12 = 0$
$(x + 4)(x - 3) = 0$
$x = 3, -4$

Area under the curve = $\int_{-4}^{3} x^2 + 2\,dx$

$= \left[\frac{1}{3}x^3 + 2x\right]_{-4}^{3}$

$= (9 + 6) - \left(\frac{-64}{3} - 8\right)$

$= 15 + \frac{88}{3} = \frac{133}{3}$ square units

Area under the line $= \int_{-4}^{3} (14 - x)\,dx$

$= \left[-\frac{(14-x)^2}{2}\right]_{-4}^{3} = \frac{-121}{2} + 162 = \frac{203}{2}$ square units

Required area $= \frac{203}{2} - \frac{133}{3}$

$= 57\frac{1}{6}$

(c) Total surface area = $x^2 + 3xh$.
$v = x^2h$
$x^2h = 0.064$

$h = \frac{0.064}{x^2}$

$\therefore A = x^2 + 3x\left(\frac{0.064}{x^2}\right)$

$= x^2 + \frac{0.192}{x}$

$\frac{dA}{dx} = 2x - \frac{0.192}{x^2}$

$\frac{dA}{dx} = 0 \Rightarrow 2x = \frac{0.192}{x^2}$

$x^3 = 0.096$
$x = 0.458$ cm

When $x = 0.458$, $A = (0.458)^2 + \frac{0.192}{0.458}$

$= 0.629\ m^2$

2 (a) (i) $\lim_{x \to \frac{5}{4}} \frac{64x^3 - 125}{12x^2 - 11x - 5}$

$= \lim_{x \to \frac{5}{4}} \frac{\cancel{(4x-5)}(16x^2 + 20x + 25)}{\cancel{(4x-5)}(3x+1)}$

$= \lim_{x \to \frac{5}{4}} \left[\frac{16x^2 + 20x + 25}{3x + 1}\right]$

$= \frac{16\left(\frac{5}{4}\right)^2 + 20\left(\frac{5}{4}\right) + 25}{3\left(\frac{5}{4}\right) + 1}$

$= 15.79$

(ii) $\lim_{x \to 0} \left(\frac{\sin 9x}{x}\right)$

$= \lim_{x \to 0} 9\left(\frac{\sin 9x}{9x}\right)$

$= 9 \lim_{x \to 0} \left(\frac{\sin 9x}{9x}\right)$

$=9(1)$

$= 9$

(b) (i) $\frac{dy}{dx} = 3x - \frac{4}{x^2}$

$y = \int \left(3x - \frac{4}{x^2}\right) dx$

$y = \frac{3}{2}x^2 + \frac{4}{x} + c$

$x=1,\ y = \frac{5}{2} \Rightarrow \frac{5}{2} = \frac{3}{2} + 4 + c$

$c = -3$

$\therefore y = \frac{3}{2}x^2 + \frac{4}{x} - 3$

(ii) At stationary points $\frac{dy}{dx} = 0$

$\Rightarrow\ 3x - \frac{4}{x^2} = 0$

$3x^3 = 4$

$x^3 = \frac{4}{3}$

$x = \sqrt[3]{\frac{4}{3}} = 1.1$

$y = \frac{3}{2}(1.1)^2 + \frac{4}{1.1} - 3$

$= 2.45$

$\frac{d^2y}{dx^2} = 3 + \frac{8}{x^3}$

$x^3 = 4/3,\ \frac{d^2y}{dx^2} = 3 + 8/_{4/3} = 6 + 3 = 9 > 0$

$\therefore$ min point at (1.1, 2.45)

(c) (i) Find $\frac{d^2y}{dx^2} = 4x^3 + 3x^2$

$\Rightarrow \frac{dy}{dx} = \int 4x^3 + 3x^2\ dx$

$\frac{dy}{dx} = x^4 + x^3 + A$

Integrating again wrt x:

$y = \int x^4 + x^3 + A\ dx$

$y = \frac{1}{5}x^5 + \frac{1}{4}x^4 + Ax + B$

(ii) When x = 0, y = 1 $\Rightarrow$ 1 = B

$\frac{dy}{dx} = x^4 + x^3 + A$

$x = 0,\ \frac{dy}{dx} = 0 \Rightarrow 0 = A$

$\therefore$ solution is

$y = \frac{1}{5}x^5 + \frac{1}{4}x^4 + 1$

3 (a) (i) $\frac{dy}{dt} \propto (9 - y)^{1/3}$

$\Rightarrow \frac{dy}{dt} = k(9 - y)^{1/3}$

when y = 1, $\frac{dy}{dt} = 0.2$

$\Rightarrow 0.2 = k(8)^{1/3}$

$k = \frac{0.2}{2}$

= 0.1

$\therefore \frac{dy}{dt} = 0.1(9 - y)^{1/3}$

(ii) $\frac{dy}{dt} = 0.1(9 - y)^{1/3}$

$\Rightarrow \int \frac{1}{(9 - y)^{1/3}} = dy \int 0.1\,dt$

$\Rightarrow \int (9 - y)^{-1/3} = dy \int 0.1\,dt$

$\Rightarrow \frac{(9 - y)^{2/3}}{-2/3} = 0.1t + c$

when t = 0, y = 1 $\Rightarrow -\frac{3}{2}(8)^{2/3} = c$

c = –6

$\therefore -\frac{3}{2}(9-y)^{2/3} = 0.1t - 6$

$(9-y)^{2/3} = -\frac{1}{15}t + 4 \Rightarrow 9 - y = \left(4 - \frac{1}{15}t\right)^{3/2}$

$y = 9 - (4 - \frac{1}{15}t)^{3/2}$

(iii) At max height $\frac{dy}{dt} = 0 \Rightarrow 9 - y = 0 \Rightarrow y = 9$

When y = 9, t = 15(4) = 60 yrs

(b) $f'(x) = 4x^3 + 6x^2 + 2x + k$

$f(x) = \int 4x^3 + 6x^2 + 2x + k\, dx$

$f(x) = x^4 + 2x^3 + x^2 + kx + c$

$f(0) = 5 \Rightarrow 5 = c$

$f(1) = 10 \Rightarrow 10 = 1 + 2 + 1 + k + 5$

$k = 1$

$\therefore f(x) = x^4 + 2x^3 + x^2 + x + 5$

(c) $y = A\cos3x + B\sin3x$

$\frac{dy}{dx} = -3A\sin3x + 3B\cos3x$

$\frac{d^2y}{dx^2} = -9A\cos3x - 9B\sin3x$

$\frac{d^2y}{dx^2} + 9y = \cancel{-9A\cos3x} - \cancel{9B\sin3x} + \cancel{9A\cos3x} + \cancel{9B\sin3x}$

$= 0$

Hence $\frac{d^2y}{dx^2} + 9y = 0$

4 (a) (i) $y = x^3 + bx^2 + x + c$

$\frac{dy}{dx} = 3x^2 + 2bx$

$\frac{d^2y}{dx^2} = 6x + 2b$

when x = 1, $\frac{d^2y}{dx^2} = 0 \Rightarrow 0 = 6 + 2b$

$b = -3$

when x = 1, y = 4

$\Rightarrow 4 = 1 - 3 + 1 + c$

$c = 5$

$b = -3,\ c = 5$

(ii) when x = 1, $\frac{dy}{dx} = 3 - 6 = -3$

Equation of the tangent: $y - 4 = -3(x - 1)$

$y + 3x = 7$

(b) $\int_0^1 \frac{x^2 + 2}{(3x^3 + 18x + 1)^3}\, dx$

$u = 3x^3 + 18x + 1$

$\frac{du}{dx} = 9x^2 + 18$

$du = 9(x^2 + 2)dx$

$\frac{1}{9} du = (x^2 + 2)\, dx$

When x = 1, u = 3 + 18 + 1 = 22

x = 0, u = 1

$$\therefore \int_0^1 \frac{x^2 + 2}{(3x^3 + 18x + 1)^3} dx = \int_1^{22} \frac{\frac{1}{9}du}{u^3} = \frac{1}{9}\int_1^{22} u^{-3}\, du = -\frac{1}{18}\left[\frac{1}{u^2}\right]_1^{22}$$

$$= -\frac{1}{18}\left[\frac{1}{22^2} - 1\right] = 0.055$$

(c) (i) $\int_0^{\pi/6} \cos 4\theta \cos 2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/6} \cos 6\theta + \cos 2\theta\, d\theta$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 6\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} = \frac{1}{2}\left[\frac{1}{6}\sin \pi + \frac{1}{2}\sin\frac{\pi}{3}\right] = \frac{\sqrt{3}}{8}$$

(ii) $\int_0^{\pi/3} \cos^2 3x\, dx = \frac{1}{2}\int_0^{\pi/3} 1 + \cos 6x\, dx$

$$= \frac{1}{2}\left[x + \frac{1}{6}\sin 6x\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{6}\sin 2\pi - 0\right]$$

$$= \frac{\pi}{6}$$

Module 3 Test 2

1 (a) point of intersection:

$-x^2 + 6x + 3 = 2x + 6$

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1, 3$

Area under the curve = $\int_1^3 -x^2 + 6x + 3\, dx$

$$= \left[-\frac{1}{3}x^3 + 3x^2 + 3x\right]_1^3$$

$$= (-9 + 27 + 9) - \left(-\frac{1}{3} + 3 + 3\right)$$

$$= 21\frac{1}{3} \text{ square units}$$

Area under the line = $\int_1^3 (2x+6)\,dx$

$= \left[x^2 + 6x\right]_1^3$

= (9 + 18) – (1 + 6)

= 20 square units

$\therefore$ shaded area $= 21\frac{1}{3} - 20 = 1\frac{1}{3}$ square units

(b) $V = \pi\int_a^b y^2 dx$

$V = \pi\int_0^{\pi/8} \cos^2(2x)dx$

$= \pi\int_0^{\pi/8} \frac{1+\cos 4x}{2}\,dx$

$= \frac{\pi}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\pi/8}$

$= \frac{\pi}{2}\left[\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2}\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{8} + \frac{1}{4}\right]$

(c) (i) $\frac{d}{dx}\left[\frac{x}{x^2+4}\right]$

$= \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2}$

$= \frac{-x^2+4}{(x^2+4)^2}$

Since $\frac{d}{dx}\left[\frac{x}{x^2+4}\right] = \frac{4 - x^2}{(x^2+4)^2}$

$\Rightarrow \left[\frac{x}{x^2+4}\right]_0^2 = \int_0^2 \frac{4-x^2}{(x^2+4)^2}\,dx$

$\Rightarrow \frac{2}{8} = \int_0^2 \frac{4\text{-}x^2}{(x^2+1)^2}\,dx$

$\times 3 \Rightarrow \frac{6}{8} = \int_0^2 \frac{12-3x^2}{(x^2+4)^2}\,dx = \frac{3}{4}$

(ii) $\int_0^2 \left[kx^3 - 2f(x)\right]\,dx = \left[\frac{k}{4}x^4\right]_0^2 - \frac{1}{2}(12)$

= 4k – 6

4k – 6 = 1

4k = 7

$k = \frac{7}{4}$

2 (a) $\frac{dv}{dt} = 300\ \pi\ cm^3s^{-1}$

(i) RTF $\frac{dr}{dt}$ when r = 25cm

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$v = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

when r = 25, $\frac{dv}{dr} = 4\pi(25)^2$

$\therefore 300\pi = 4\pi(25)^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{\pi 300}{4\pi (25)^2} = \frac{3}{25}\ cms^{-1}$

(ii) RTF $\frac{dA}{dt}$ when r = 25cm

Solution:

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

when r = 25, $\frac{dA}{dr} = 25(8\pi) = 200\pi$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 200\pi \times \frac{3}{25} = 24\pi\ cm^2s^{-1}$

(b) (i) $y = x + \frac{6}{x^2}$

$\frac{dy}{dx} = 1 - \frac{12}{x^3}$

$\frac{d^2y}{dx^2} = \frac{36}{x^4}$

$x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x\left(\frac{36}{x^4}\right) + 3\left[1 - \frac{12}{x^3}\right]$

$= \frac{36}{x^3} - \frac{36}{x^3} + 3$

$\Rightarrow x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 3$

(ii) when x = 1, $\frac{dy}{dx} = 1 - 12 = -11$

Gradient of the normal = $\frac{1}{11}$

x = 1, y = 1 + 6 = 7.

Equation of the normal at (1, 7) is

$y - 7 = \frac{1}{11}(x - 1)$

$11y - 77 = x - 1$

$11y - x = 76$

(c) (i) $y = \sin x^2$

$\frac{dy}{dx} = 2x \cos x^2$

Since $\frac{d}{dx}\left[\sin x^2\right] = 2x \cos x^2$

$\Rightarrow \left[\sin x^2\right]_0^{\pi/2} = \int_0^{\pi/2} 2x \cos x^2 dx$

$\Rightarrow \frac{1}{2}\sin\left(\frac{\pi^2}{4}\right) = \int_0^{\pi/2} x \cos x^2 dx$

$= 0.312$

(ii) $x^3 \frac{dy}{dx} = x + 1$

$\frac{dy}{dx} = \frac{x+1}{x^3}$

$\frac{dy}{dx} = \frac{1}{x^2} + \frac{1}{x^3}$

$\Rightarrow y = \int \frac{1}{x^2} + \frac{1}{x^3} dx$

$y = -\frac{1}{x} - \frac{1}{2x^2} + c$

3 (a) (i) $\frac{d}{dx}\left[(x^2 + 2) \tan x\right] = (x^2 + 2)\sec^2 x + 2x \tan x$

(ii) $\frac{d}{dx}\left[\cos(5x^3 - 2x)^{1/2}\right] = \frac{1}{2}(5x^3 - 2x)^{-1/2}(15x^2 - 2)(-\sin(5x^3 - 2x)^{1/2})$

$= \frac{2 - 15x^2}{2\sqrt{5x^3 - 2x}} \sin(\sqrt{5x^3 - 2x})$

(b) $f(-2) = -2a + b$

$-2a + b = -2$ [1]

$f(2) = 2a + b$

$2a + b = \frac{1}{2}(2) - 1$

$2a + b = 0$ [2]

[1] + [2] $\Rightarrow 2b = -2, b = -1$

$a = \frac{1}{2}$

$a = \frac{1}{2}, b = -1$

(c) (i) $\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4}$

$= \lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4} \times \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3}$

$$= \lim_{x \to 4} \frac{x + 5 - 9}{(x - 4)\sqrt{x + 5} + 3}$$

$$= \lim_{x \to 4} \frac{\cancel{x - 4}}{\cancel{x - 4}\,(\sqrt{x + 5} + 3)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x + 5} + 3}$$

$$= \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

(ii) $\lim_{x \to 0} \left(\frac{\tan 2x}{x}\right)$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$= \lim_{x \to 0} \left(\frac{1}{\cos 2x}\right) \times \lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)$$

$= 1 \times (2)$
$= 2$

4 (a) $y = x^3 - 6x^2 + 9x + 1$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x + 9 = 0$$

$x^2 - 4x + 3 = 0$
$(x - 1)(x - 3) = 0$
$x = 1, 3$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 12 = 0$$

$x = 2$

when $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0$ Max

$x = 3$, $\frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0$ Min

$x = 1, y = 1 - 6 + 9 + 1 = 5$
$x = 2, y = 8 - 24 + 18 + 1 = 3$
$x = 3, y = 27 - 54 + 27 + 1 = 1$
$\therefore$ (1, 5) Maximum point
(2, 3) point of inflexion
(3, 1) Minimum point

(b) $\int_0^p (x - 2)^3\,dx = 0$

$$\Rightarrow \left[\frac{(x - 2)^4}{4}\right]_0^p = 0$$

$\Rightarrow \dfrac{(p-2)^4}{4} - 4 = 0$

$(p - 2)^4 = 16$

$p - 2 = \sqrt[4]{16}$

$p = 2 + 2$

$= 4$

(c) (i) $y = \dfrac{2x+1}{x-4}$

Vertical asymptote : $x - 4 = 0$

$x = 4$

Horizontal asymptote :

$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\dfrac{2x+1}{x-4}\right)$

$= \lim_{x\to\infty} \dfrac{2+\frac{1}{x}}{1-\frac{4}{x}}$

$= \dfrac{2+0}{1-0} = 2$

$\therefore$ y = 2 is a horizontal asymptote

Asymptotes are: x = 4, vertical

y = 2, horizontal

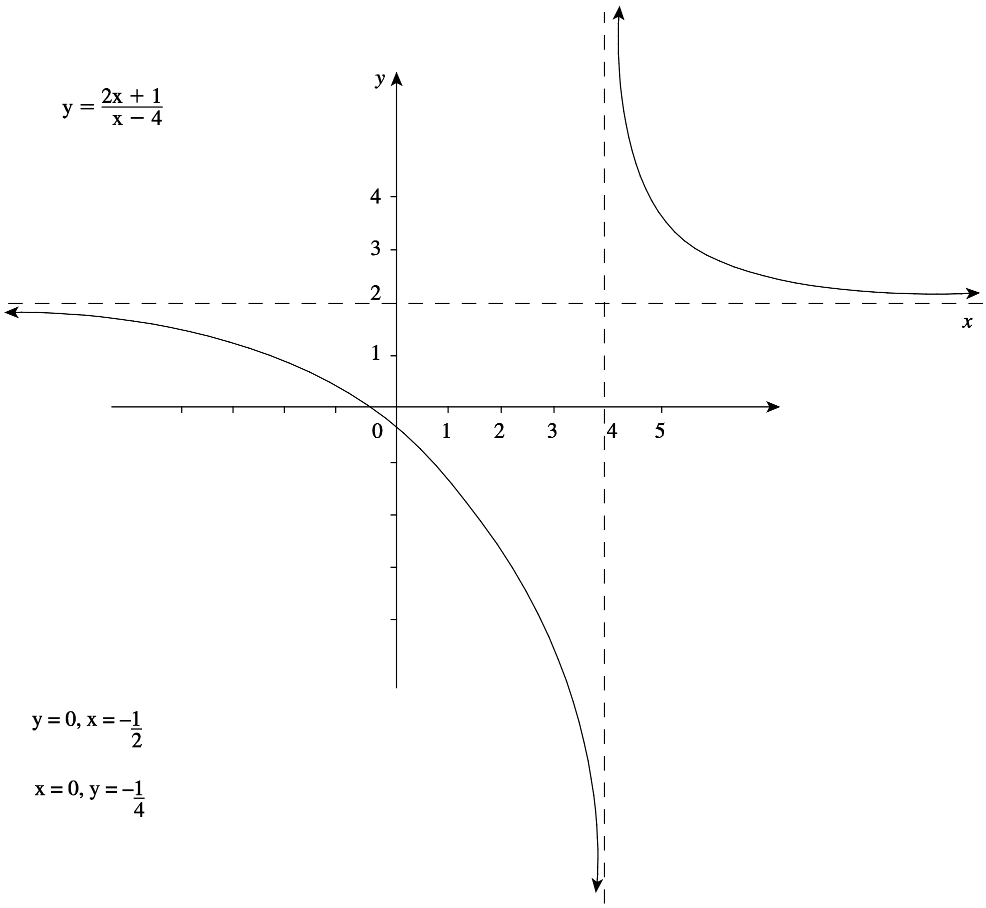
(ii) $\dfrac{dy}{dx} = \dfrac{(x-4)(2)-(2x+1)(1)}{(x-4)^2}$

$= \dfrac{2x-8-2x-1}{(x-4)^2}$

$= \dfrac{-9}{(x-4)^2}$

$\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{-9}{(x-4)^2} = 0 \Rightarrow -9 = 0$, inconsistent

$\therefore$ There are no turning points


y = (2x + 1)/(x − 4)
y
4
3
2
1
0
1
2
3
4
5
x
y = 0, x = −1/2
x = 0, y = −1/4