Chapter 7 Cubic Polynomials

Review Exercise 7

1 (a) $2x^2 = 3x - 5$

$2x^2 - 3x + 5 = 0$

$$\alpha + \beta = \frac{3}{2}$$

$$\alpha\beta = \frac{5}{2}$$

$$\frac{\alpha}{2\beta+1} + \frac{\beta}{2\alpha+1}$$

$$= \frac{\alpha(2\alpha+1) + \beta(2\beta+1)}{(2\beta+1)(2\alpha+1)}$$

$$= \frac{2\alpha^2 + \alpha + 2\beta^2 + \beta}{4\alpha\beta + 2\beta + 2\alpha + 1}$$

$$= \frac{2(\alpha^2 + \beta^2) + \alpha + \beta}{4\alpha\beta + 2(\alpha+\beta) + 1}$$

$$= \frac{2[(\alpha+\beta)^2 - 2\alpha\beta] + (\alpha+\beta)}{4\alpha\beta + 2(\alpha+\beta) + 1}$$

$$= \frac{2\left(\left(\frac{3}{2}\right)^2 - 2\left(\frac{5}{2}\right)\right) + \frac{3}{2}}{4\left(\frac{5}{2}\right) + 2\left(\frac{3}{2}\right) + 1}$$

$$= \frac{\frac{-11}{2} + \frac{3}{2}}{10 + 3 + 1} = \frac{-4}{14} = \frac{-2}{7}$$

(b) Sum of the roots $= \frac{\alpha}{2\beta+1} + \frac{\beta}{2\alpha+1} = \frac{-2}{7}$

Product of roots $= \left(\frac{\alpha}{2\beta+1}\right)\left(\frac{\beta}{2\alpha+1}\right)$

$$= \frac{\alpha\beta}{(2\beta+1)(2\alpha+1)} = \frac{\frac{5}{2}}{14} = \frac{5}{28}$$

The equation is:

$$x^2 + \frac{2}{7}x + \frac{5}{28} = 0$$

$28x^2 + 8x + 5 = 0$

2 $2x^2 - 5x + 7 = 0$

$$\alpha + \beta = \frac{5}{2}$$

$$\alpha\beta = \frac{7}{2}$$

Sum of the roots $= \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta}$

$= (\alpha + \beta) + \frac{\beta + \alpha}{\alpha\beta}$

$= \frac{5}{2} + \frac{\frac{5}{2}}{\frac{7}{2}}$

$= \frac{5}{2} + \frac{5}{7} = \frac{45}{14}$

Product of the roots $= \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

$= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$

$= \alpha\beta + \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{1}{\alpha\beta}$

$= \alpha\beta + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta}$

$= \frac{7}{2} + \frac{\left(\frac{5}{2}\right)^2 - 2\left(\frac{7}{2}\right)}{\frac{7}{2}} + \frac{1}{\frac{7}{2}}$

$= \frac{7}{2} + \left(-\frac{3}{14}\right) + \frac{2}{7}$

$= \frac{25}{7}$

The equation with roots $\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta}$ is

$x^2 - \frac{45}{14}x + \frac{25}{7} = 0$

$14x^2 – 45x + 50 = 0$

3 (a) $3x^2 = – (2x – 1)$

$\Rightarrow 3x^2 + 2x – 1 = 0$

$\alpha + \beta = \frac{-2}{3}$

$\alpha\beta = -\frac{1}{3}$

$\alpha^2 + \beta^2 + 2\alpha\beta = (\alpha + \beta)^2 = \left(\frac{-2}{3}\right)^2 = \frac{4}{9}$

(b) $\alpha^4 - \beta^4 + \alpha^2 - \beta^2$

$= (\alpha^2 - \beta^2)(\alpha^2 + \beta^2) + \alpha^2 - \beta^2$

$= (\alpha^2 - \beta^2)(\alpha^2 + \beta^2 + 1)$

$= (\alpha - \beta)(\alpha + \beta)((\alpha + \beta)^2 - 2\alpha\beta + 1)$

$$= \left(\frac{-2}{3}\right)\left\{\left(\frac{-2}{3}\right)^2 - 2\left(-\frac{1}{3}\right) + 1\right\}(\alpha - \beta)$$

$$= \frac{-38}{27}(\alpha - \beta)$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \left(\frac{-2}{3}\right)^2 - 4\left(-\frac{1}{3}\right) = \frac{4}{9} + \frac{4}{3} = \frac{16}{9}$$

$$\alpha - \beta = \sqrt{\frac{16}{9}} = \frac{4}{3},\ \alpha > \beta$$

$$\therefore\ \alpha^4 - \beta^4 + \alpha^2 - \beta^2 = \frac{-38}{27} \times \frac{4}{3} = \frac{-152}{81}$$

4 (a) $x^3 - 10x + 6 = 0$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = -10$

$\alpha\beta\gamma = -6$

(b) $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$= (0)^2 - 2(-10)$

$= 20.$

(c) $x = \alpha \Rightarrow \alpha^3 - 10\alpha + 6 = 0$

$x = \beta \Rightarrow \beta^3 - 10\beta + 6 = 0$

$x = \gamma \Rightarrow \gamma^3 - 10\gamma + 6 = 0$

Add] $\Rightarrow \sum\alpha^3 - 10\sum\alpha + 18 = 0$

$\sum\alpha^3 - 10(0) + 18 = 0$

$\therefore\ \sum\alpha^3 = -18.$

5 (a) $2x^3 - x^2 - 10x - 6 = 0$

$$\sum\alpha = \alpha + \beta + \gamma = \frac{1}{2}$$

$$\sum\alpha\beta = -5$$

$$\alpha\beta\gamma = \frac{-(-6)}{2} = 3$$

$$\sum\alpha = \frac{1}{2}$$

(b) $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$$= \left(\frac{1}{2}\right)^2 - 2(-5)$$

$$= 10\frac{1}{4}$$

(c)
$$\left.\begin{array}{l} x = \alpha,\ 2\alpha^3 - \alpha^2 - 10\alpha - 6 = 0 \\ x = \beta,\ 2\beta^3 - \beta^2 - 10\beta - 6 = 0 \\ x = \gamma,\ 2\gamma^3 - \gamma^2 - 10\gamma - 6 = 0 \end{array}\right\}\text{add}$$

$$\Rightarrow 2\sum\alpha^3 - \sum\alpha^2 - 10\sum\alpha - 18 = 0$$

$$\therefore\ 2\sum\alpha^3 - \left(10\frac{1}{4}\right) - 10\left(\frac{1}{2}\right) - 18 = 0$$

$$\sum \alpha^3 = \frac{1}{2}\left(18 + 5 + 10\frac{1}{4}\right)$$

$$= \frac{133}{8}$$

6 (a) $x^3 + 4x + 1 = 0$

Since the roots are p, q and r

$p + q + r = 0$

$pq + pr + qr = 4$

$pqr = -1$

$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + pr + qr)$

$= (0)^2 - 2(4)$

$= -8$

(b) $p^3 + 4p + 1 = 0$

$q^3 + 4q + 1 = 0$

$r^3 + 4r + 1 = 0$

$p^3 + q^3 + r^3 + 4(p + q + r) + 3 = 0$

$p^3 + q^3 + r^3 + 4(0) + 3 = 0$

$\therefore p^3 + q^3 + r^3 = -3$

7 $2x^3 - 4x^2 + 6x - 1 = 0$

$$\alpha + \beta + \gamma = \frac{-(-4)}{2} = 2$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{6}{2} = 3$$

$$\alpha\beta\gamma = \frac{1}{2}$$

Sum of the roots $= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{3}{\frac{1}{2}} = 6$$

Sum of product of pairs

$$= \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) + \left(\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}\right) + \left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)$$

$$= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma} = \frac{2}{\frac{1}{2}} = 4$$

Product of the roots $= \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)$

$$= \frac{1}{\alpha\beta\gamma}$$

$$= \frac{1}{\frac{1}{2}} = 2$$

The equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

$x^3 - 6x^2 + 4x - 2 = 0$
OR:

Let $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Substituting $x = \frac{1}{y}$ into the equation

$2\left(\frac{1}{y}\right)^3 - 4\left(\frac{1}{y}\right)^2 + 6\left(\frac{1}{y}\right) - 1 = 0$

$\times\, y^3 \Rightarrow 2 - 4y + 6y^2 - y^3 = 0$

$\therefore\ y^3 - 6y^2 + 4y - 2 = 0$

Since $x = \alpha, \beta, \gamma \Rightarrow y = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

The equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

$y^3 - 6y^2 + 4y - 2 = 0$

8 The roots of the equation are – 2, – 3 and 4

$\therefore\ (x + 2)(x + 3)(x - 4) = 0$

$\Rightarrow (x^2 + 5x + 6)(x - 4) = 0$

$\Rightarrow x^3 - 4x^2 + 5x^2 - 20x + 6x - 24 = 0$

$\Rightarrow x^3 + x^2 - 14x - 24 = 0$

The equation with roots – 2, – 3 and 4 is

$x^3 + x^2 - 14x - 24 = 0$

Comparing with $x^3 + \alpha x^2 + \beta x + \gamma = 0$

$\Rightarrow \alpha = 1, \beta = -14, \gamma = -24$

OR:

Sum of the roots = (– 2) + (– 3) + 4

= – 1

Sum of the product of pairs = (– 2) (– 3) + (– 2) (4) + (– 3) (4)

= 6 – 8 – 12 = – 14

Product of the roots = (– 2) (– 3) (4) = 24.

The equation is

$x^3 + x^2 - 14x - 24 = 0$

$\therefore\ \alpha = 1, \beta = -14, \gamma = -24$

9 (a) $x^3 + 6x^2 + 10x + 14 = 0$

$\alpha + \beta + \gamma = -6, \quad \alpha\beta + \alpha\gamma + \beta\gamma = 10, \quad \alpha\beta\gamma = -14$

Let $y = x^2 \Rightarrow x = \sqrt{y}$

Substituting into the equation:

$(\sqrt{y})^3 + 6(\sqrt{y})^2 + 10\sqrt{y} + 14 = 0$

$y\sqrt{y} + 6y + 10\sqrt{y} + 14 = 0$

$\sqrt{y}\,(y + 10) = -14 - 6y$

Squaring: $y(y + 10)^2 = (-14 - 6y)^2$

$y(y^2 + 20y + 100) = 196 + 168y + 36y^2$

$y^3 - 16y^2 - 68y - 196 = 0$

Hence the equation with roots α^2, β^2 and γ^2 is

$y^3 - 16y^2 - 68y - 196 = 0$

(b) Let $y = x + 3$

$\Rightarrow x = y - 3$

Substituting into the equation:

$(y-3)^3 + 6(y-3)^2 + 10(y-3) + 14 = 0$

$\Rightarrow y^3 - 9y^2 + 27y - 27 + 6(y^2 - 6y + 9) + 10y - 30 + 14 = 0$

$\Rightarrow y^3 - 3y^2 + y + 11 = 0$

The equation with roots $\alpha + 3, \beta + 3, \gamma + 3$ is

$y^3 - 3y^2 + y + 11 = 0$

OR:

Sum of the roots $= \alpha + 3 + \beta + 3 + \gamma + 3$

$= (\alpha + \beta + \gamma) + 9$

$= -6 + 9 = 3$

Sum of the product of pairs $= (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$

$= \alpha\beta + 3\alpha + 3\beta + 9 + \alpha\gamma + 3\alpha + 3\gamma + 9 + \beta\gamma + 3\beta + 3\gamma + 9$

$= (\alpha\beta + \alpha\gamma + \beta\gamma) + 6(\alpha + \beta + \gamma) + 27$

$= 10 + 6(-6) + 27$

$= 1$

Product of the roots

$= (\alpha + 3)(\beta + 3)(\gamma + 3)$

$= (\alpha\beta + 3\alpha + 3\beta + 9)(\gamma + 3)$

$= \alpha\beta\gamma + 3\alpha\beta + 3\alpha\gamma + 9\alpha + 3\beta\gamma + 9\beta + 9\gamma + 27$

$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha + \beta + \gamma) + 27$

$= -14 + 3(10) + 9(-6) + 27$

$= -11$

The equation with roots $\alpha + 3, \beta + 3, \gamma + 3$ is

$x^3 - 3x^2 + x + 11 = 0$

10 $3x^3 - 4x^2 + 8x - 7 = 0$

Let $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$\therefore\ 3\left(\frac{1}{y}\right)^3 - 4\left(\frac{1}{y}\right)^2 + 8\left(\frac{1}{y}\right) - 7 = 0$

$\Rightarrow 3 - 4y + 8y^2 - 7y^3 = 0$

The equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

$7y^3 - 8y^2 + 4y - 3 = 0$

OR:

$3x^3 - 4x^2 + 8x - 7 = 0$

$\sum \alpha = \frac{4}{3}$

$\sum \alpha\beta = \frac{8}{3}$

$\sum \alpha\beta\gamma = \frac{7}{3}$

Sum of the roots $= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \frac{\frac{8}{3}}{\frac{7}{3}} = \frac{8}{7}$

Sum of the product of pairs $= \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) + \left(\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}\right) + \left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)$

$= \dfrac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \dfrac{\frac{4}{3}}{\frac{7}{3}} = \dfrac{4}{7}$

Product of roots $= \left(\dfrac{1}{\alpha}\right)\left(\dfrac{1}{\beta}\right)\left(\dfrac{1}{\gamma}\right)$

$= \dfrac{1}{\alpha\beta\gamma} = \dfrac{1}{\frac{7}{3}} = \dfrac{3}{7}$

The equation with roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ is

$x^3 - \dfrac{8}{7}x^2 + \dfrac{4}{7}x - \dfrac{3}{7} = 0$

$7x^3 - 8x^2 + 4x - 3 = 0$

11 (a) $2x^3 - x^2 - 10x - 6 = 0$

$\alpha + \beta + \gamma = \dfrac{1}{2}$

(b) $\alpha^2 + \beta^2 + \gamma^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$= \left(\dfrac{1}{2}\right)^2 - 2(-5)$

$= 10\dfrac{1}{4}$

(c) $9\beta^2 + 9\gamma^2 = 32\alpha^2$

$\beta^2 + \gamma^2 = \dfrac{32}{9}\alpha^2$

Substitute into $\alpha^2 + \beta^2 + \gamma^2 = 10\dfrac{1}{4}$

$\Rightarrow \alpha^2 + \dfrac{32}{9}\alpha^2 = \dfrac{41}{4}$

$\dfrac{41}{9}\alpha^2 = \dfrac{41}{4} \Rightarrow \alpha^2 = \dfrac{9}{4}, \alpha = \dfrac{-3}{2}, \alpha < 0$

$\alpha + \beta + \gamma = \dfrac{1}{2} \Rightarrow \beta + \gamma = \dfrac{1}{2} + \dfrac{3}{2} = 2$

$\gamma = 2 - \beta$

$\alpha\beta\gamma = 3 \Rightarrow \left(-\dfrac{3}{2}\right)\beta(2 - \beta) = 3$

$2\beta - \beta^2 = -2$

$\beta^2 - 2\beta - 2 = 0$

$\beta = \dfrac{2 \pm \sqrt{4 + 8}}{2}$

$= \dfrac{2 \pm \sqrt{12}}{2}$

$= 1 \pm \sqrt{3}$

$\beta > 0 \therefore \beta = 1 + \sqrt{3}, \ \gamma = 1 - \sqrt{3}$

$$\alpha = \frac{-3}{2}, \beta = 1 + \sqrt{3}, \gamma = 1 - \sqrt{3}$$

12 $x^3 + ax^2 + bx + c = 0$

$\alpha + \beta + \gamma = -a$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

$\alpha\beta\gamma = -c$

$\alpha + \beta + \gamma = 6 \Rightarrow -a = 6$

$a = -6$

$\sum \alpha^2 = 14$

$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

$= (-6)^2 - 2b$

$36 - 2b = 14$

$2b = 22$

$b = 11$

$\sum \alpha^3 + a\sum \alpha^2 + b\sum \alpha + 3c = 0$

$36 + (-6)(14) + (11)(+6) + 3c = 0$

$36 - 84 + 66 + 3c = 0$

$c = 6$

$\therefore\ a = -6, b = 11, c = 6$

13 $x^3 + ax^2 + bx + c = 0$

$\sum \alpha = -a, \sum \alpha\beta = b, \alpha\beta\gamma = -c$

Now $\sum \alpha = 0 \Rightarrow a = 0$

$\sum \alpha^2 = 14 \Rightarrow \left(\sum \alpha\right)^2 - 2\sum \alpha\beta = 14$

$-2b = 14$

$b = -7$

$\sum \alpha^3 + a\sum \alpha^2 + b\sum \alpha + 3c = 0$

$18 + 0 - 7(0) + 3c = 0$

$c = -6$

$a = 0, b = -7, c = -6$

14 $x^3 - 6x + 3 = 0$

Let $y = \frac{x+1}{x}$

$\Rightarrow xy = x + 1$

$\Rightarrow xy - x = 1$

$x(y - 1) = 1$

$x = \frac{1}{y-1}$

Substituting into the equation:

$$\left(\frac{1}{y-1}\right)^3 - 6\left(\frac{1}{y-1}\right) + 3 = 0$$

$\times (y-1)^3 \Rightarrow 1 - 6(y-1)^2 + 3(y-1)^3 = 0$

$1 - 6[y^2 - 2y + 1] + 3[y^3 - 3y^2 + 3y - 1] = 0$

$\therefore\ 3y^3 - 15y^2 + 21y - 8 = 0$

$\therefore$ The equation with roots $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}, \frac{\gamma+1}{\gamma}$ is

$3x^3 - 15x^2 + 21x - 8 = 0$

15 (a) $x^3 - 2x^2 + 4x + 5 = 0$

$\alpha + \beta + \gamma = 2$

$\alpha\beta + \beta\gamma + \gamma\alpha = 4$

$\alpha\beta\gamma = -5$

Sum of the roots $= 2\alpha + 2\beta + 2\gamma$

$= 2(\alpha + \beta + \gamma)$

$= 2(2) = 4$

Sum of the product of pairs $= (2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma)$

$= 4[\alpha\beta + \alpha\gamma + \beta\gamma]$

$= 4(4) = 16$

Product of the roots $= (2\alpha)(2\beta)(2\gamma)$

$= 8\alpha\beta\gamma = 8(-5) = -40$

The equation is

$x^3 - 4x^2 + 16x + 40 = 0$

OR:

$y = 2x \Rightarrow x = \frac{y}{2} \quad \therefore \left(\frac{y}{2}\right)^3 - 2\left(\frac{y}{2}\right)^2 + 4\left(\frac{y}{2}\right) + 5 = 0$

$\times 8 \Rightarrow y^3 - 4y^2 + 16y + 40 = 0$

(b) $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3 - 2\left(\frac{1}{y}\right)^2 + 4\left(\frac{1}{y}\right) + 5 = 0$

$\times y^3 \Rightarrow 1 - 2y + 4y^2 + 5y^3 = 0$

The equation with roots

$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

$5x^3 + 4x^2 - 2x + 1 = 0$