Chapter 6 Functions

Try these 6.1

(a) This relation is a function.
(b) This relation is not a function since e does not map onto any values in the range.
(c) $1 \rightarrow 7$ and $1 \rightarrow 8$, Since 1 maps onto two different values, this relation is not a function.

Try these 6.2

(a) $x - 4$ must be positive in order to find a real solution for $\sqrt{(x-4)}$
$\therefore x - 4 \geq 0,\ x \geq 4$
Hence, domain is $x: x \in \mathbb{R},\ x \geq 4$
Range is $y \in \mathbb{R}$
(b) $x + 1 > 0,\ x > -1$
Hence, domain is $x: x \in \mathbb{R},\ x > -1$
Range is $y \in \mathbb{R}$
(c) Domain is $x \in \mathbb{R}$
As $x \rightarrow \infty,\ e^x \rightarrow \infty,\ y \rightarrow \infty$
As $x \rightarrow -\infty,\ e^x \rightarrow 0,\ y \rightarrow 1$
Range is $y: y \in \mathbb{R}, y > 1$

Exercise 6A

1 (a) This is not a function since $7 \rightarrow 8$ and $7 \rightarrow 9$
i.e. the value in the domain maps onto two different values in the range.
(b) This relation is a function and the mapping is:
$-1 \rightarrow 2$
$1 \rightarrow 2$
$-3 \rightarrow 8$
$3 \rightarrow 8$
(c) Not a function since c maps onto two different values.
(d) This relation is a function and the mapping is:
$1 \rightarrow 2$
$2 \rightarrow 4$
$3 \rightarrow 6$
$4 \rightarrow 8$

2 (a) This is a function {(a,1), (b,1), (c,2), (d,3)}
(b) This is not a function since t maps onto two different values.
(c) This is a function since each value in the domain maps onto one value in the range.
{(2, a), (4, b), (6, c), (8, d), (10, c), (12, b)}

3 $g(x) = 4x - 5$
(a) $x = 0$, $g(x) = -5$
$x = 1$, $g(x) = 4 - 5 = -1$
$x = 2$, $g(x) = 8 - 5 = 3$

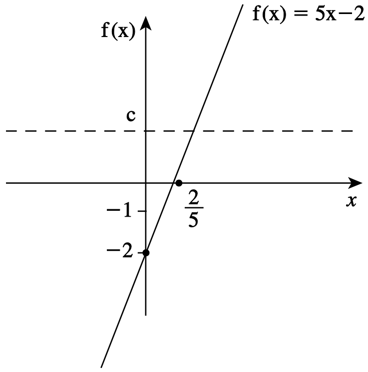
x = 3, g(x) = 12 – 5 = 7
x = 4, g(x) = 16 – 5 = 11
Range of g is {–5, –1, 3, 7, 11}

(b) {(0, – 5), (1, –1), (2, 3), (3, 7) (4, 11)}

4 $f(x) = \begin{cases} x + 2 & \text{if } x < 2 \\ x - 1 & \text{if } x > 8 \\ (x+1)^2 & \text{if } 2 \le x \le 8 \end{cases}$

(a) f (–4) = – 4 + 2 = –2
(b) f (9) = 9 – 1 = 8
(c) f (2) = $(2 + 1)^2$ = 9
(d) f (8) = $(8 + 1)^2$ = 81

5 f(x) = 5x – 2, $x \in \mathbb{R}$



(a) Any line y = c drawn parallel to the x-axis will cut the graph only once.
⇒ f (x) is one-to-one.

(b) Any line drawn parallel to the x-axis cuts the graph at least once (exactly once in this case): for every y there is a corresponding x mapping onto it. Hence f(x) is an onto function.

6 $f(x) = \dfrac{1}{x-4}$

$f(a) = \dfrac{1}{a-4}$

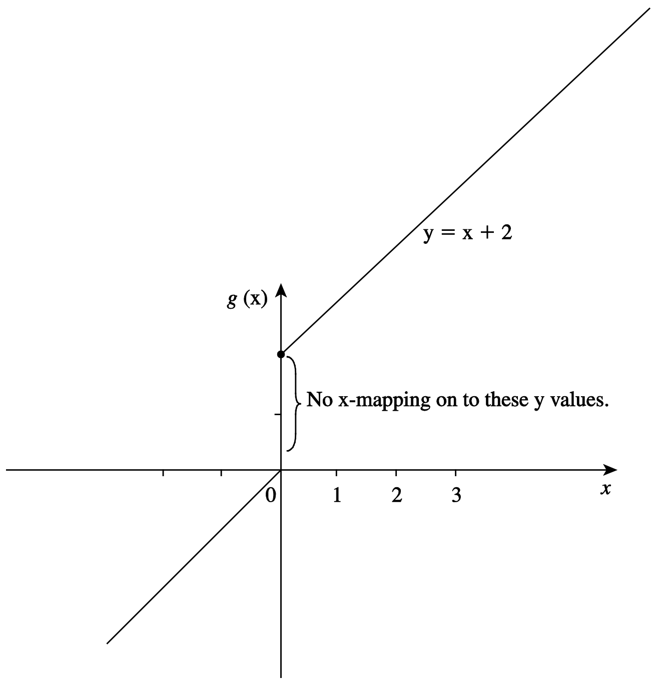
$f(b) = \dfrac{1}{b-4}$

f (a) = f (b)

$\Rightarrow \dfrac{1}{a-4} = \dfrac{1}{b-4}$

⇒ a – 4 = b – 4
a= b
Since f (a) = f (b) ⇒ a = b
f (x) is one-to-one or injective

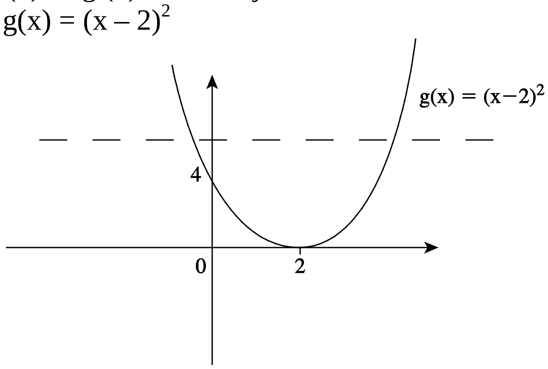
7 $g: \mathbb{R} \to \mathbb{R}$
Let us draw the graph of g (x) and use the graph to answer both (a) and (b)
g (x) = x + 2, x ≥ 0
g (x) = x, x < 0



(a) g (x) is injective since every x maps onto one and only one y

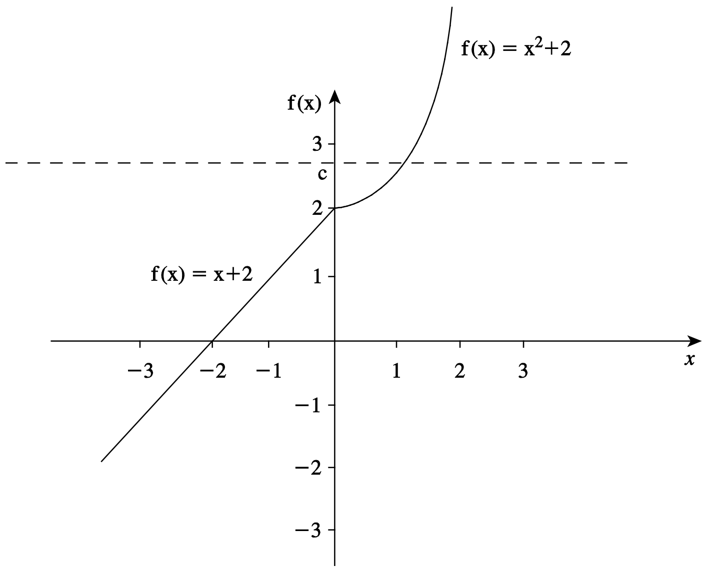
(b) g (x) is not surjective since there are no x-values mapping onto the y-values from 0 to 2

8 $g(x) = (x - 2)^2$



Any line drawn parallel to the x-axis will cut the graph at least once for $y \in \mathbb{R}^+$
Hence every y has a corresponding x mapping onto. Therefore g (x) is surjective.

9 (a) $f(x) = x^2 + 2, x \geq 0$
$x + 2, x < 0$



Any line drawn parallel to the x-axis cuts the graph once.
Hence f(x) is one-to-one and onto
$\therefore$ f (x) is bijective

(b) Let $y = x^2 + 2$
To find the inverse, let
$x = y^2 + 2$
$y^2 = x - 2$
$y = \pm\sqrt{x-2}$
Since the range of $f(x) \geq 0$,
$f^{-1}(x) = \sqrt{x-2}$ for $x \geq 2$
Let $y = x + 2$
$x = y + 2$
$y = x - 2$
$$\therefore f^{-1}(x) = \begin{cases} \sqrt{x-2}, & x \geq 2 \\ x - 2, & x < 2 \end{cases}$$

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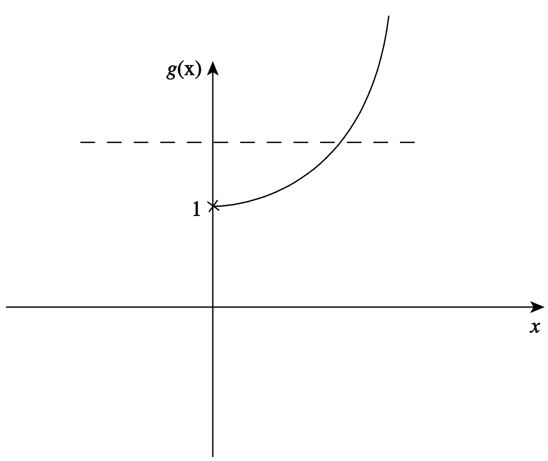
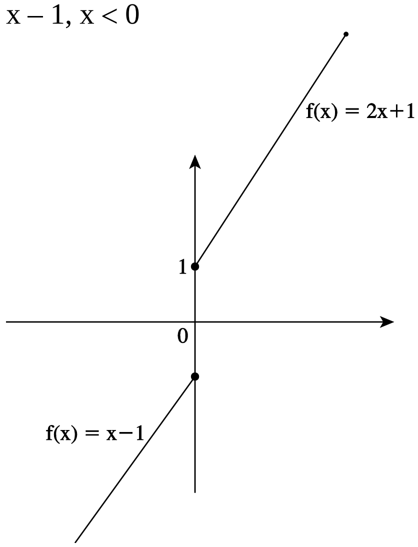


(a) Since a line drawn parallel to the x-axis cuts the graph at most once $\Rightarrow$ g(x) is one-to-one.
(b) When y = 0, there is no x-value mapping onto it

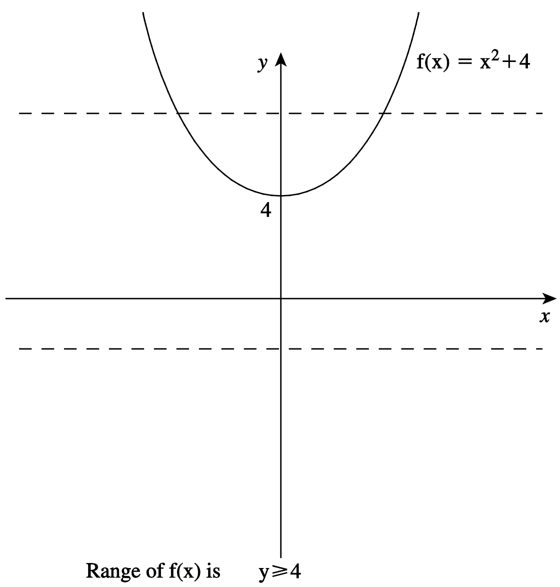
Hence y is not surjective.

11 f (x) = 2x + 1, $x \geq 0$
x – 1, $x < 0$



From the graph there are no x-values mapping onto y from –1 to 1 Hence f (x) is not surjective. For f (x) to be bijective , f (x) must be both injective and surjective.
∴ f (x) is not bijective.

12 (a)



(b) Any line drawn parallel to the x-axis for y > 4 will cut the graph twice.
Hence f (x) is not injective.
Any line drawn parallel to the x-axis for y < 4 will not cut the graph, hence f (x) is not surjective.

(c) g: $x \to x^2 + 4$, $x \geq 0$, $f(x) \geq 4$

(d) $y = x^2 + 4$
$x = y^2 + 4$
$y^2 = x - 4$
$y = \sqrt{x - 4}$

Hence $g^{-1}(x) = \sqrt{x - 4}$, $x \geq 4$

13 (a) $\alpha \to 1$ and $\alpha \to 4$, since the same x-value maps onto two different y-values, g is not a function. Also μ has no value assigned to it.

(b) Removing the mapping $\alpha \to 4$, and assigning $\mu \to 3$
we have f: $\{(\alpha, 1), (\beta, 2), (\delta, 3), (\varepsilon, 4), (\theta, 2), (\mu, 3)\}$

Try these 6.2

(a) This relation is a function.
(b) This relation is not a function since e does not map onto any values in the range.
(c) $1 \to 7$ and $1 \to 8$, Since 1 maps onto two different values, this relation is not a function.

Exercise 6B

1 $f(x) = 4x - 2$
$g(x) = 6x + 1$
$gf = g(4x - 2)$ gf: $x \to 24x - 11$
$= 6(4x - 2) + 1$
$= 24x - 11$
$fg = f(6x + 1)$ fg: $x \to 24x + 2$
$= 4(6x + 1) - 2$
$= 24x + 2$

2 $f(x) = 3x + 5$
$g(x) = 2x^2 + x + 1$
$gf = g(3x + 5)$ gf: $x \to 18x^2 - 63x + 56$
$= 2(3x + 5)^2 + 3x + 5 + 1$
$= 2[9x^2 + 30x + 25] + 3x + 6$
$= 18x^2 + 63x + 56$
$fg = f(2x^2 + x + 1)$ fg : $x \to 6x^2 + 3x + 8$
$= 3(2x^2 + x + 1) + 5$
$= 6x^2 + 3x + 8$

3 $f(x) = x + 4$
$g(x) = \frac{2}{x}$
$gf = g(x + 4) = \frac{2}{x + 4}$
$fg = f\left(\frac{2}{x}\right) = \frac{2}{x} + 4$
gf: $x \to \frac{2}{x + 4}$, $x \neq -4$
fg: $x \to \frac{2}{x} + 4$, $x \neq 0$

4 $f(x) = 1 + 5x$

$g(x) = \dfrac{x+1}{x-1}$

$gf = g(1 + 5x)$

$= \dfrac{1+5x+1}{1+5x-1}$

$= \dfrac{5x+2}{5x}$

$fg = f\left(\dfrac{x+1}{x-1}\right)$

$= 1 + 5\left(\dfrac{x+1}{x-1}\right)$

$= \dfrac{x-1+5x+5}{x-1}$

$= \dfrac{6x+4}{x-1}$

$gf: x \to \dfrac{5x+2}{5x}, x \neq 0$

$fg: x \to \dfrac{6\,x+4}{x-1}, x \neq 1$

5 $g(x) = 2x - 1$

$g^2 = g\,(2x-1) = 2(2x-1) - 1 = 4x - 3$

$g^3 = gg^2 = g\,(4x-3) = 2(4x-3) - 1 = 8x - 7$

6 $g(x) = \dfrac{x}{2x+1}$

$$g^2 = g\left(\dfrac{x}{2x+1}\right) = \dfrac{\dfrac{x}{2x+1}}{\dfrac{2x}{2x+1}+1}$$

$$= \dfrac{\dfrac{x}{2x+1}}{\dfrac{2x+2x+1}{2x+1}}$$

$$= \dfrac{x}{4x+1}, x \neq -\dfrac{1}{2}, x \neq -\dfrac{1}{4}$$

$$g^3 = g\left(\dfrac{x}{4x+1}\right) = \dfrac{\dfrac{x}{4x+1}}{\dfrac{2x}{4x+1}+1} = \dfrac{\dfrac{x}{4x+1}}{\dfrac{2x+4x+1}{4x+1}} = \dfrac{x}{6x+1},\ x \neq \dfrac{-1}{2},\ x \neq \dfrac{-1}{4},\ x \neq \dfrac{-1}{6}$$

7 $g(x) = \dfrac{2x+1}{x+1}$

$g^2 = g\left(\dfrac{2x+1}{x+1}\right)$

$$= \frac{2\left(\frac{2x+1}{x+1}\right)+1}{\frac{2x+1}{x+1}+1}$$

$$= \frac{\frac{4x+2+x+1}{x+1}}{\frac{2x+1+x+1}{x+1}}$$

$$= \frac{5x+3}{x+1} \times \frac{x+1}{3x+2}$$

$$= \frac{5x+3}{3x+2}, x \neq -1, x \neq -\frac{2}{3}$$

$$g^3 = g\left(\frac{5x+3}{3x+2}\right)$$

$$= \frac{2\left(\frac{5x+3}{3x+2}\right)+1}{\frac{5x+3}{3x+2}+1}$$

$$= \frac{\frac{10x+6+3x+2}{3x+2}}{\frac{5x+3+3x+2}{3x+2}}$$

$$= \frac{13x+8}{8x+5}, x \neq -1, x \neq -\frac{2}{3}, x \neq -\frac{5}{8}.$$

8 $g(x) = \frac{3}{4x-2}$

$$g^2 = g\left(\frac{3}{4x-2}\right)$$

$$= \frac{3}{\frac{12}{4x-2}-2}$$

$$= \frac{3}{\frac{12-8x+4}{4x-2}}$$

$$= \frac{3(4x-2)}{16-8x}$$

$$= \frac{6(2x-1)}{8(2-x)}$$

$$= \frac{3(2x-1)}{4(2-x)}$$

$g^3 = g\left(\dfrac{3(2x-1)}{4(2-x)}\right)$

$= \dfrac{3}{\dfrac{6x-3}{2-x} - 2}$

$= \dfrac{3(2-x)}{6x-3-4+2x}$

$= \dfrac{3(2-x)}{8x-7}$

9 $g(x) = x^2 + 2x + 3$

$h(x) = x - 2$

$gh = g(x-2)$

$= (x-2)^2 + 2(x-2) + 3$

$= x^2 - 4x + 4 + 2x - 4 + 3$

$= x^2 - 2x + 3$

$hg = h(x^2 + 2x + 3)$

$= x^2 + 2x + 3 - 2$

$= x^2 + 2x + 1$

Now gh = hg

$\Rightarrow \cancel{x^2} - 2x + 3 = \cancel{x^2} + 2x + 1$

$2 = 4x$

$x = \dfrac{1}{2}$

10 $f(x) = 3x - 4$

$g(x) = \dfrac{x}{x+2}$

$fg = f\left(\dfrac{x}{x+2}\right) = \dfrac{3x}{x+2} - 4$

$= \dfrac{3x-4x-8}{x+2}$

$= \dfrac{-x-8}{x+2}$

$gf = g(3x-4)$

$= \dfrac{3x-4}{3x-4+2} = \dfrac{3x-4}{3x-2}$

fg = gf

$\Rightarrow \dfrac{-x-8}{x+2} = \dfrac{3x-4}{3x-2}$

$\Rightarrow (-x-8)(3x-2) = (3x-4)(x+2)$

$\Rightarrow -3x^2 + 2x - 24x + 16 = 3x^2 + 6x - 4x - 8$

$6x^2 + 24x - 24 = 0$

$x^2 + 4x - 4 = 0$

$(x-2)^2 = 0, x = 2$

11 (a) $y = 4x - 3$

$x = 4y - 3$

$\frac{x+3}{4} = y$

$\therefore f^{-1}: x \to \frac{x+3}{4}$

(b) $y = \frac{5}{x-2}$

$x = \frac{5}{y-2}$

$xy - 2x = 5$

$y = \frac{5+2x}{x}$

$\therefore f^{-1}: x \to \frac{5+2x}{x}, x \neq 0$

(c) $y = \frac{3x-1}{x+2}$

$x = \frac{3y-1}{y+2}$

$\Rightarrow xy + 2x = 3y - 1$

$\Rightarrow xy - 3y = -1 - 2x$

$y(x-3) = -1 - 2x$

$y = \frac{-1-2x}{x-3}$

$= \frac{2x+1}{3-x}$

$\therefore f^{-1}: x \to \frac{2x+1}{3-x}, x \neq 3$

12 (a) $4x^2 + 12x + 3 = a(x+b)^2 + c$

$= ax^2 + 2abx + ab^2 + c.$

Equating coefficients:

$\Rightarrow a = 4$

$2ab = 12 \Rightarrow 8b = 12, b = \frac{12}{8} = \frac{3}{2}$

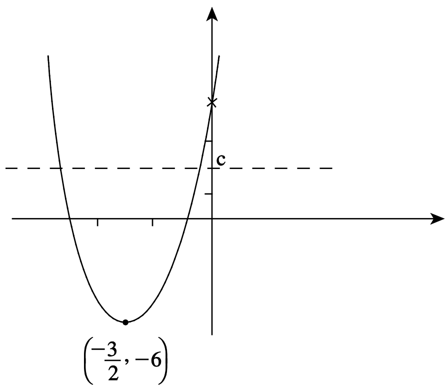
$ab^2 + c = 3 \Rightarrow 4\left(\frac{3}{2}\right)^2 + c = 3, c = -6$

Hence

$4x^2 + 12x + 3 = 4\left(x + \frac{3}{2}\right)^2 - 6$

The range of the function is $y \geq -6$

(b)



A line y = c drawn parallel to the x-axis cuts the graph more than once hence the function is not one-to-one and therefore has no inverse.

13 (a) $f(x) = \frac{4x-1}{3x+2}$

Let $y = \frac{4x-1}{3x+2}$

$x = \frac{4y-1}{3y+2}$

$x(3y + 2) = 4y - 1$

$3xy + 2x = 4y - 1$

$3xy - 4y = -2x - 1$

$y(3x - 4) = -2x - 1$

$y = \frac{-2x-1}{3x-4}$

$= \frac{1+2x}{4-3x}$

$\therefore f^{-1}(x) = \frac{1+2x}{4-3x}, x \neq \frac{4}{3}$

$f^{-1}(1) = \frac{1+2}{4-3} = 3$

$f^{-1}(-1) = \frac{1-2}{4+3} = -\frac{1}{7}$

(b) $f^{-1}(x) = x$

$\Rightarrow \frac{1+2x}{4-3x} = x$

$\Rightarrow 1 + 2x = x(4 - 3x)$

$\Rightarrow 1 + 2x = 4x - 3x^2$

$\Rightarrow 3x^2 - 2x + 1 = 0$

$x = \frac{2 \pm \sqrt{4-12}}{6}$

$= \frac{2 \pm \sqrt{-8}}{6}$

Since $\sqrt{-8}$ is not real

$\Rightarrow$ there is no real x for which

$f^{-1}(x) = x$

14 $f(x) = \dfrac{4x}{x+1}$, $g(x) = x + 4$

(a) $k = -1$

(b) $fg = f(x + 4)$

$$= \frac{4(x+4)}{x+4+1}$$

$$= \frac{4x+16}{x+5}$$

fg is not defined when $x = -5$

(c) Let $y = \dfrac{4x}{x+1}$

$$x = \frac{4y}{y+1}$$

$xy + x = 4y$

$xy - 4y = -x$

$y(x - 4) = -x$

$$y = \frac{-x}{x-4} = \frac{x}{4-x}$$

$$f^{-1}(x) = \frac{x}{4-x}, x \neq 4$$

(d) $f^{-1}(a) = \dfrac{a}{4-a}$, $g(-1) = 3$

$$\frac{a}{4-a} = 3 \Rightarrow a = 12 - 3a, 4a = 12, a = 3$$

Try these 6.3

(a) $y = \dfrac{x+1}{x-2}$

$$\begin{array}{r} 1 \\ x-2\,\overline{)\,x+1} \\ \underline{x-2} \\ 3 \end{array}$$

$$\therefore y = 1 + \frac{3}{x-2}$$

$$f(x) = \frac{1}{x}$$

$f(x-2) = \frac{1}{x-2}$

$3\,f(x-2) = \frac{3}{x-2}$

$1 + 3\,f(x-2) = 1 + \frac{3}{x-2}$

$\frac{1}{x}$ is shifted to the right by 2 units and stretched along the y-axis by factor 3. This graph is then moved upwards by 1 unit.

(b) $y = \frac{2x+1}{x+1}$

$$\begin{array}{r} 2 \\ x+1\overline{)2x+1} \\ \underline{2x+2} \\ -1 \end{array}$$

$\therefore\ y = 2 - \frac{1}{x+1}$

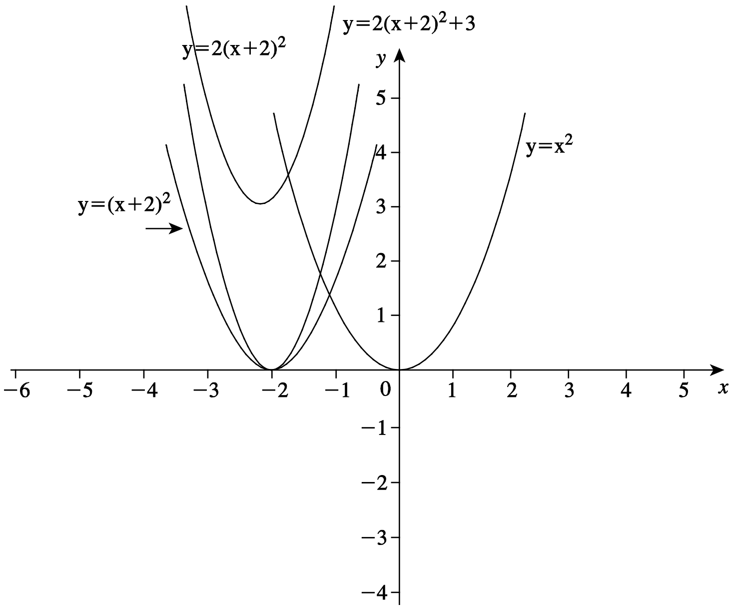
$f(x) = \frac{1}{x} \Rightarrow f\,(x+1)\ \frac{1}{x+1} \Rightarrow -f\,(x+1) = -\frac{1}{x+1}$

$\Rightarrow 2 - f(x+1) = 2 - \frac{1}{x+1}$

$\therefore\ \frac{1}{x}$ is shifted to the left by 1 unit, then reflected in the x-axis and moved upwards by 2 units.

Exercise 6c

1



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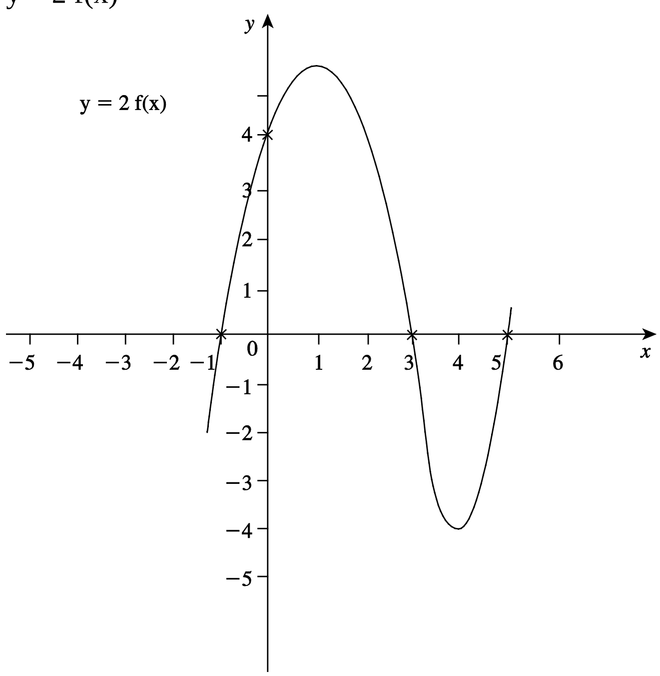
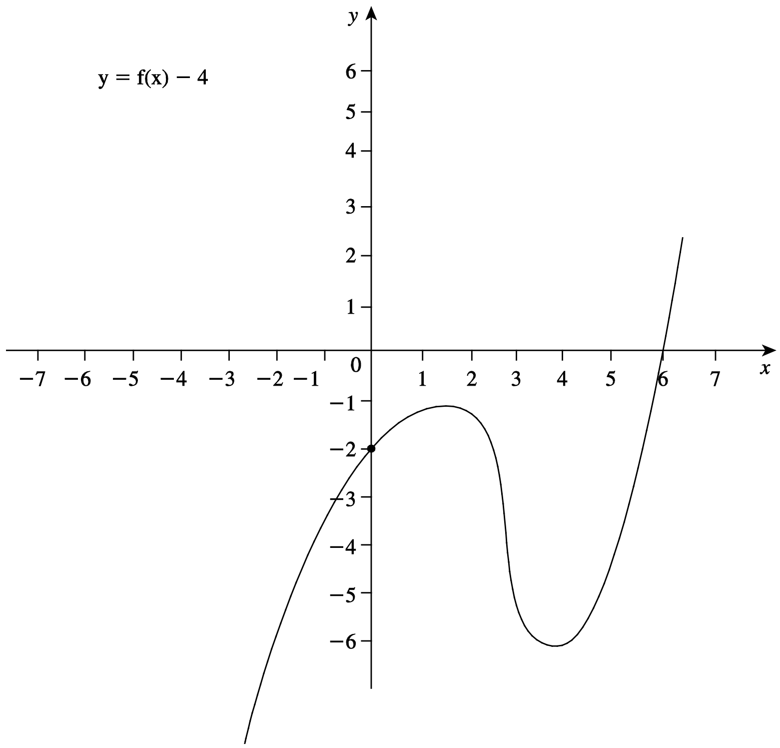
y = 2 f(x)



y = f(x) – 4



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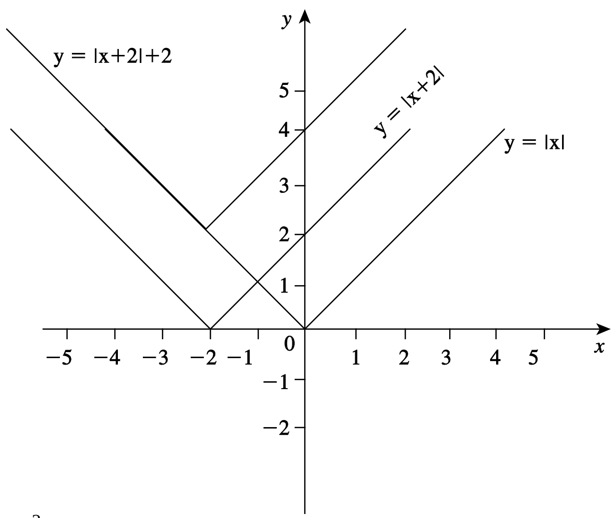


4 $2x^2 - 4x + 6$

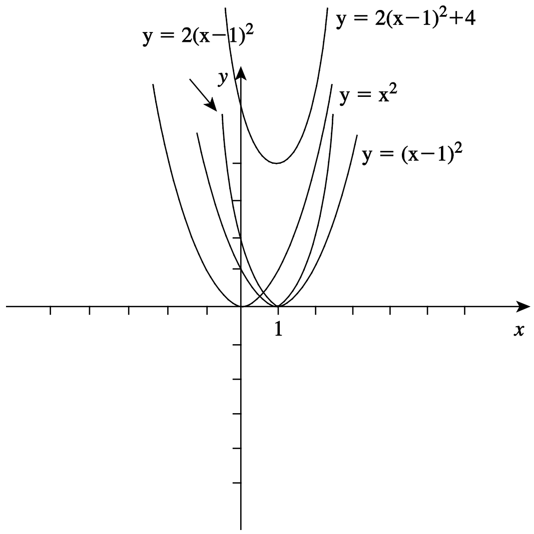
$= 2(x^2 - 2x) + 6$

$= 2\ (x - 1)^2 + 6 - 2$

$= 2(x - 1)^2 + 4$

$y = 2(x - 1)^2 + 4$

Starting with $y = x^2$, shift to the right by 1 unit, then stretch along the y-axis by factor 2 and up the y-axis by 4 units.



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$$\begin{array}{r} 1 \\ x+2 \overline{)\, x+1} \\ \underline{x+2} \\ -1 \end{array}$$

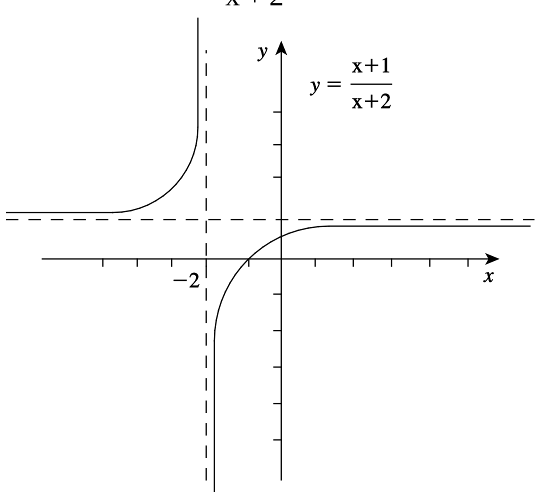
$\therefore\ y = 1 - \dfrac{1}{x+2}$

$f(x) = \dfrac{1}{x}$

$f(x+2) = \dfrac{1}{x+2}$

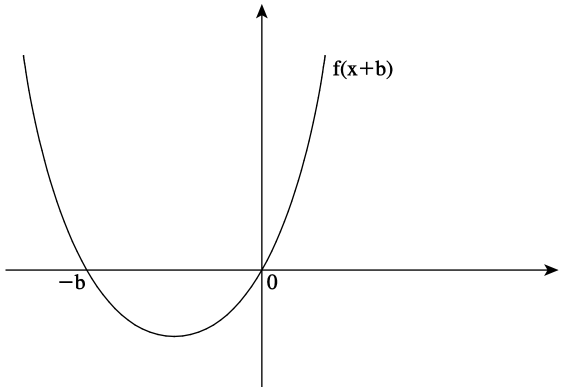
$-\,f(x+2) = -\,\dfrac{1}{x+2}$

$1 - f(x+2) = 1 - \dfrac{1}{x+2}$

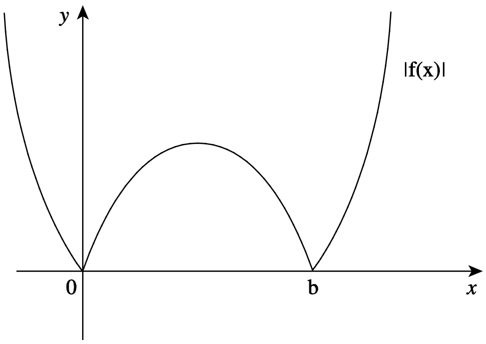


Transformations are: shift $\frac{1}{x}$ to the left by 2 units followed by a reflection in the x-axis and a translation upwards by 1 unit.

6 (a)



(b)



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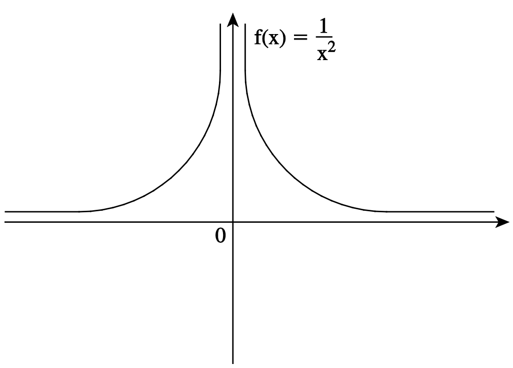
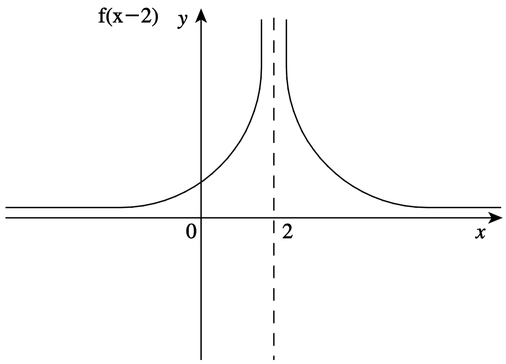
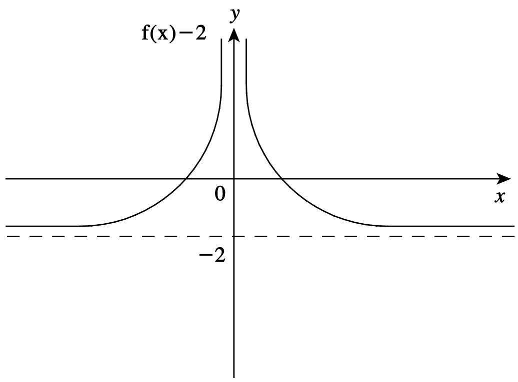


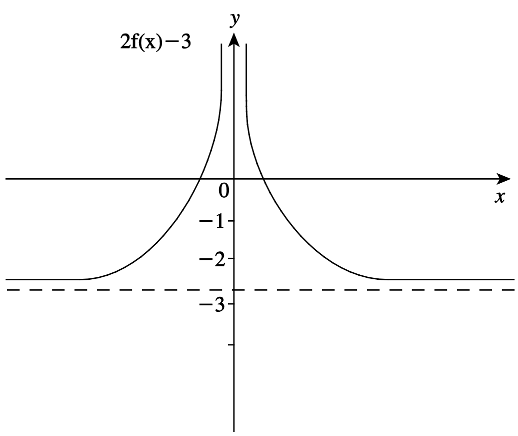
(a)



(b)



(c)



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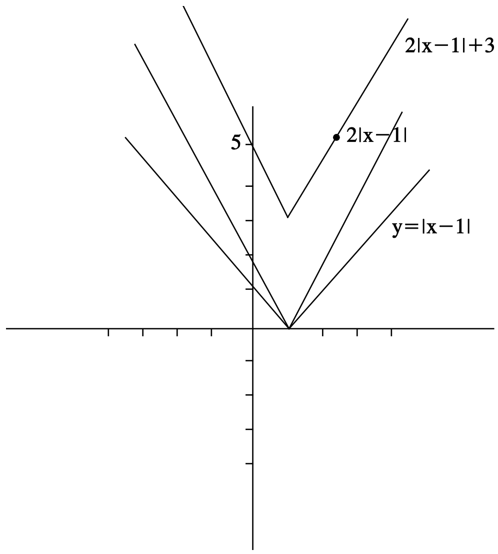


9

$$\begin{array}{r} 3 \\ x+2\overline{)3x+1} \\ \underline{3x+6} \\ -5 \end{array}$$

$$y = 3 - \frac{5}{x+2}$$

Shift $\frac{1}{x}$ to the left by 2 units, reflect in the x-axis and stretch by factor 5 units along the y-axis finally move upwards by 3 units.

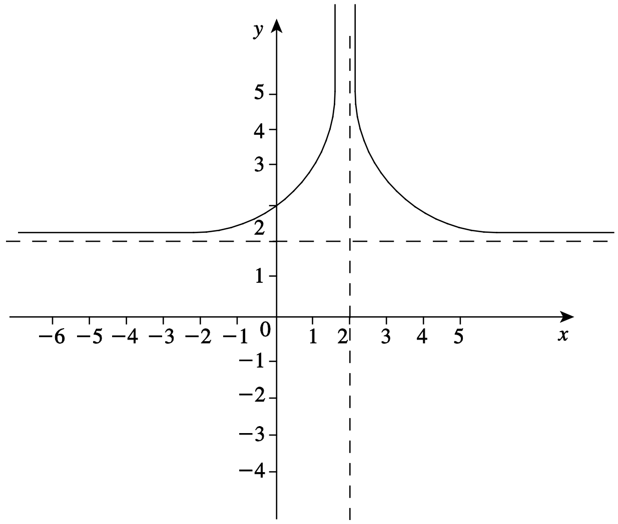
10 $y = \dfrac{2x^2 - 8x + 9}{x^2 - 4x + 4}$

$$\begin{array}{r} 2 \\ x^2-4x+4\overline{)2x^2-8x+9} \\ \underline{2x^2-8x+8} \\ 1 \end{array}$$

$$y = 2 + \frac{1}{x^2 - 4x + 4}$$

$$= 2 + \frac{1}{(x-2)^2}$$

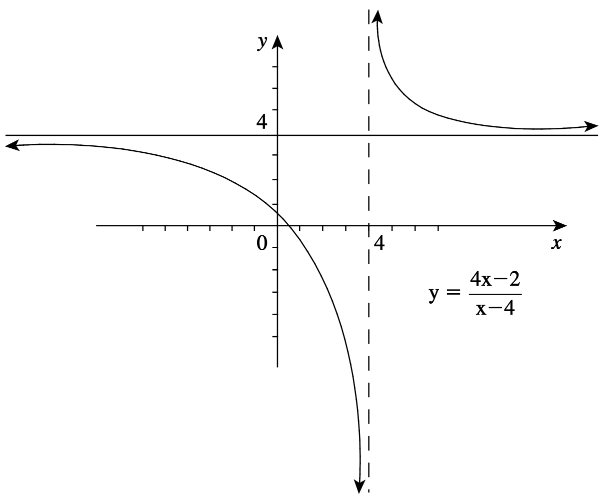
Shift $\frac{1}{x}$ to the right by 2 units move upwards by 2 units



11 (a) $y = \dfrac{4x-2}{x-4}$

$$\begin{array}{r} 4 \\ x-4\overline{)4x-2} \\ \underline{4x-16} \\ 14 \end{array}$$

$y = 4 + \dfrac{14}{x-4}$

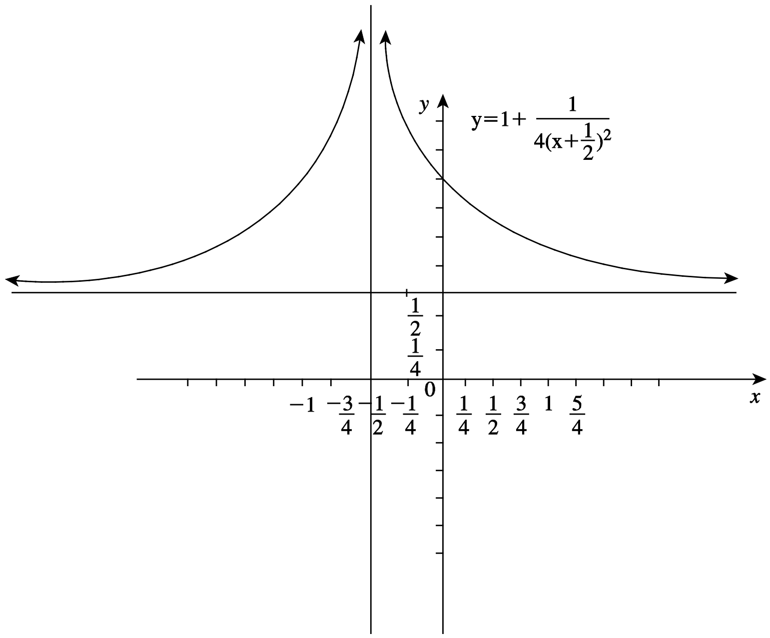


(b) $y = \dfrac{4x^2+4x+2}{4x^2+4x+1}$

$$\begin{array}{r} 1 \\ 4x^2+4x+1\overline{)4x^2+4x+2} \\ \underline{4x^2+4x+1} \\ 1 \end{array}$$

$y = 1 + \dfrac{1}{4x^2+4x+1}$

$$= 1 + \frac{1}{(2x+1)^2} = 1 + \frac{1}{4\left(x+\frac{1}{2}\right)^2}$$



Review Exercise 6

1 (a) Function
(b) Not a function, since a maps onto two different values and d has no correspondence
(c) Function
(d) Function
(e) Not a function since β has no corresponding mapping.
(f) Function

2 (a) Function
(b) Function
(c) Not a function since $6 \to 7$ and $6 \to 8$. Every value in the domain must map onto one value in the range.
(d) Not a function since $c \to d$ and $c \to e$

3 $f(x) = 4x - 3$

$f(4) = 16 - 3 = 13$

$f(-3) = -12 - 3 = -15$

$f\left(\frac{1}{2}\right) = 2 - 3 = -1$

$f\left(-\frac{1}{8}\right) = -\frac{1}{2} - 3 = -\frac{7}{2}$

$\therefore$ Images are 13, – 15, – 1, $-\frac{7}{2}$

4 $f(x) = 4 - \frac{3}{4}x$

(a) $f(1) = 4 - \frac{3}{4} = \frac{13}{4}$

(b) $f(2) = 4 - \frac{3}{2} = \frac{5}{2}$

(c) $f(0) = 4 - \frac{3}{4}(0) = 4$

(d) $f(4) = 4 - \frac{3}{4}(4) = 1$

5 $f(x) = 3x - 7$
$g(x) = 4x + 2$

(a) $f(0) + g(2) = 3(0) - 7 + 4(2) + 2$
$= 3$

(b) $2f(3) + g(1)$
$= 2[3(3) - 7] + 4(1) + 2$
$= 4 + 4 + 2 = 10$

(c) $2f(1) - 3g(2)$
$= 2[3 - 7] - 3[8 + 2]$
$= -8 - 30$
$= -38$

(d) $4f(-1) + 3g(-2)$
$= 4[-3 - 7] + 3[-8 + 2]$
$= -40 - 18$
$= -58$

6 (a) $f(x) = \frac{1}{2}x + \frac{3}{4}$

$g(x) = \frac{5}{6}x + \frac{2}{3}$

$f(x) = g(x)$

$\frac{1}{2}x + \frac{3}{4} = \frac{5}{6}x + \frac{2}{3}$

$\frac{3}{4} - \frac{2}{3} = \frac{2}{6}x$

$\frac{1}{12} = \frac{2}{6}x$

$x = \frac{1}{4}$

(b) $f(x) = \frac{1}{4}x$

$\Rightarrow \frac{1}{2}x + \frac{3}{4} = \frac{1}{4}x$

$\frac{1}{4}x = -\frac{3}{4}$

$x = -3$

(c) $f(2x) = 3g(x)$

$\frac{1}{2}(2x) + \frac{3}{4} = 3\left[\frac{5}{6}x + \frac{2}{3}\right]$

$\Rightarrow x + \frac{3}{4} = \frac{5}{2}x + 2$

$\frac{3}{4} - 2 = \frac{3}{2}x$

$\frac{-5}{4} = \frac{3}{2}x$

$x = -\frac{5}{6}$

(d) $f\left(\frac{1}{2}x\right) = g\left(\frac{1}{4}x\right)$

$\frac{1}{2}\left(\frac{1}{2}x\right) + \frac{3}{4} = \frac{5}{6}\left(\frac{1}{4}x\right) + \frac{2}{3}$

$\frac{1}{4}x + \frac{3}{4} = \frac{5}{24}x + \frac{2}{3}$

$\frac{1}{4}x - \frac{5}{24}x = \frac{2}{3} - \frac{3}{4}$

$\frac{1}{24}x = -\frac{1}{12}$

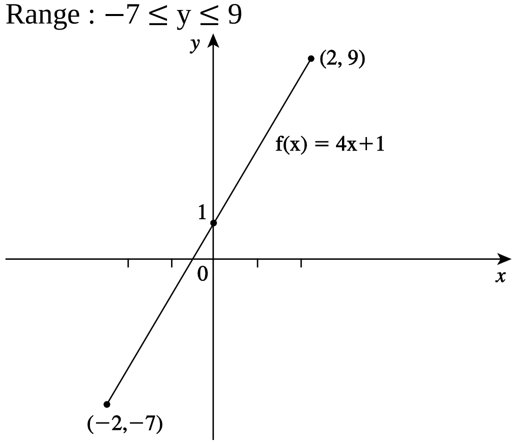
$x = -2$

7 (a) $f(x) = 4x + 1, -2 \le x \le 2$

$x = 2, f(x) = 8 + 1 = 9$

$x = -2, f(x) = -8 + 1 = -7$

Range : $-7 \le y \le 9$

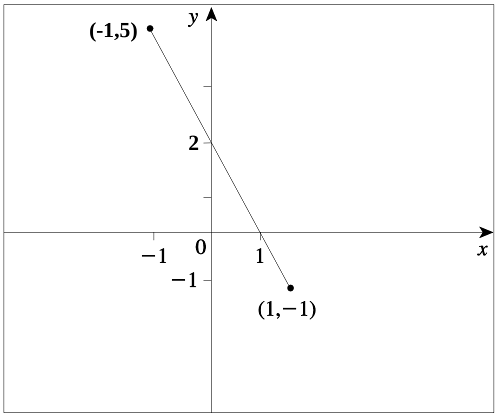


(b) $f(x) = 2 - 3x, -1 \le x \le 1$

$x = 1, f(x) = 2 - 3 = -1$

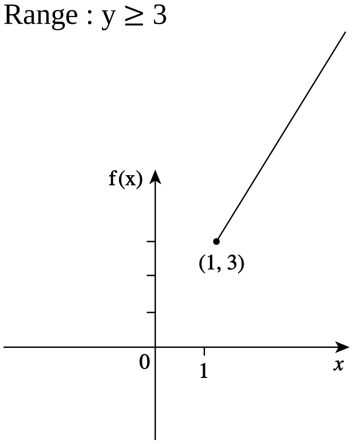
$x = -1, f(x) = 2 - 3(-1) = 5$

Range : $-1 \le y \le 5$



(c) $f(x) = 2 + x, x \geq 1$

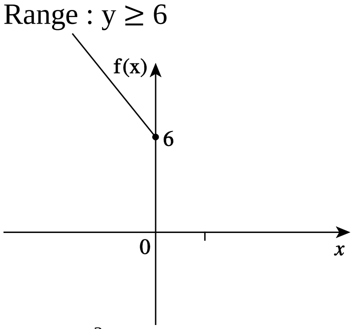
Range : $y \geq 3$



(d) $f(x) = -3x + 6, x \leq 0$

$x = 0, f(x) = 6$

Range : $y \geq 6$



8 (a) $f(x) = x^2 - 4$

Domain: $x \in \mathbb{R}$

Range: $y \geq -4$

(b) $f(x) = x^2 + 2x + 3$

$= (x + 1)^2 + 2$

Min pt. at (–1, 2)

Domain: $x \in \mathbb{R}$

Range: $y \geq 2$

(c) $f(x) = -4x^2 + x + 1$

$= -4\left(x^2 - \frac{1}{4}x\right) + 1$

$= -4\left(x - \frac{1}{8}\right)^2 + \frac{17}{16}$

Max pt. at $\left(\frac{1}{8}, \frac{17}{16}\right)$

Domain: $x \in \mathbb{R}$

Range: $y \le \frac{17}{16}$

(d) $f(x) = -(x^2 + 3x) + 5$

$= -\left(x + \frac{3}{2}\right)^2 + 5 + \frac{9}{4}$

$= -\left(x + \frac{3}{2}\right)^2 + \frac{29}{4}$

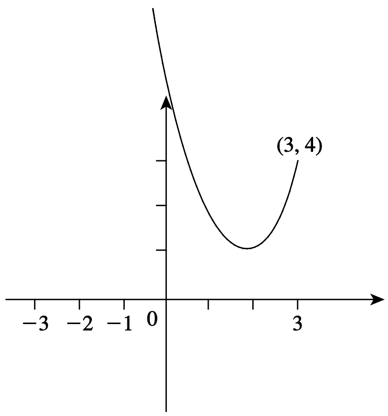
Max pt at $\left(-\frac{3}{2}, \frac{29}{4}\right)$

Domain: $x \in \mathbb{R}$

Range: $y \ge \frac{29}{4}$

9 (a) Minimum value = 1
When x = 2

(b)



10 (a) $y = 1 - \frac{2}{x}$

Rearrange to make x the subject

$1 - y = \frac{2}{x}$, $x = \frac{2}{1-y}$

Domain: $x \in \mathbb{R}, x \ne 0$

Range: $y \in \mathbb{R}, y \ne 1$

(b) $y = \frac{4x+2}{x-3}$

$y(x - 3) = 4x + 2$
$xy - 4x = 2 + 3y$
$x = \dfrac{2+3y}{y-4}$
Domain: $x \in \mathbb{R}, x \neq 3$
Range: $y \in \mathbb{R}, y \neq 4$

(c) $y = \sqrt{x-4}$
$y^2 = x - 4$
$x = y^2 + 4$
Domain: $x \in \mathbb{R}, x > 4$
Range: $y \in \mathbb{R}, y \geq 0$

11 $f(x) = 6x + 2$
$g(x) = 7x - 1$
$fg = f(7x - 1)$
$= 6(7x - 1) + 2$
$= 42x - 4$
$gf = g(6x + 2)$
$= 7(6x + 2) - 1$
$= 42x + 13$
$fg(0) = 42(0) - 4 = -4$
$fg(-2) = 42(-2) - 4 = -84 - 4 = -88$
$gf(0) = 42(0) + 13 = 13$
$gf(-2) = 42(-2) + 13 = -84 + 13 = -71$

12 $g(x) = x + 3$
$gh = x^2 + 3x - 2$
$h = g^{-1}(x^2 + 3x - 2)$
Let $y = x + 3$
$x = y + 3$
$y = x - 3$
$g^{-1}(x) = x - 3$
$\therefore h(x) = x^2 + 3x - 2 - 3$
$= x^2 + 3x - 5$

13 $fg = \dfrac{x+1}{x+2}$
$g = f^{-1}\left(\dfrac{x+1}{x+2}\right)$
$f(x) = 4x - 2$
$y = 4x - 2$
$x = 4y - 2$
$y = \dfrac{x+2}{4}$
$f^{-1}(x) = \dfrac{x+2}{4}$

$$g = \frac{\frac{x+1}{x+2}+2}{4}$$
$$= \frac{x+1+2x+4}{4(x+2)}$$
$$= \frac{3x+5}{4(x+2)}$$

14 $f(x) = \frac{3}{x+1}, x \neq -1$

$g(x) = 4x - 2$

(a) (i) $fg = f(4x - 2)$
$$= \frac{3}{4x-2+1}$$
$$= \frac{3}{4x-1}, x \neq \frac{1}{4}, x \neq -1$$

(ii) $gf = g\left(\frac{3}{x+1}\right)$
$$= 4\left(\frac{3}{x+1}\right) - 2$$
$$= \frac{12}{x+1} - 2$$
$$= \frac{12-2x-2}{x+1}$$
$$= \frac{10-2x}{x+1}, x \neq -1$$

(b) $fg = gf$

$\Rightarrow \frac{3}{4x-1} = \frac{10-2x}{x+1}$

$\Rightarrow 3(x + 1) = (10 - 2x)(4x - 1)$

$\Rightarrow 3x + 3 = 40x - 10 - 8x^2 + 2x$

$\Rightarrow 8x^2 - 39x + 13 = 0$

$b^2 - 4ac = (-39)^2 - 4(8)(13)$

$= 1105$

Since $b^2 - 4ac > 0 \Rightarrow$ there are two real and distinct solutions to fg = gf

15 $f(x) = \frac{6}{x-3}, x \neq k$

$g(x) = 5x - 3$

(a) $k = 3$

(b) (i) $gf = g\left(\frac{6}{x-3}\right)$

$= \frac{30}{x-3} - 3$

gf: $x \to \frac{30}{x-3} - 3,\ x \in \mathbb{R},\ x \neq 3$

(ii) $y = \frac{6}{x-3}$

$x = \frac{6}{y-3}$

$xy - 3x = 6$

$xy = 6 + 3x$

$y = \frac{6+3x}{x}$

$f^{-1}(x) : x \to \frac{6+3x}{x},\ x \neq 0$

(c) $y = 5x - 3$

$x = 5y - 3$

$y = \frac{x+3}{5}$

$g^{-1}(x) = \frac{x+3}{5}$

$g^{-1}(4) = \frac{7}{5}$

$fg^{-1}(4) = f\left(\frac{7}{4}\right)$

$= \frac{6}{\frac{7}{5} - 3}$

$= -\frac{15}{4}$

16 $y = \frac{4x+1}{3x-2}$

$x = \frac{4y+1}{3y-2}$

$3xy - 2x = 4y + 1$

$3xy - 4y = 2x + 1$

$y(3x - 4) = 2x + 1$

$y = \frac{2x+1}{3x-4}$

$f^{-1} : x \to \frac{2x+1}{3x-4},\ x \neq \frac{4}{3}$

(a) $f^{-1}(2) = \frac{2(2)+1}{3(2)-4} = \frac{5}{2}$

(b)

$$f\left(\frac{1}{2}\right) = \frac{4\left(\frac{1}{2}\right)+1}{3\left(\frac{1}{2}\right)-2} = \frac{3}{-\frac{1}{2}} = -6$$

$$ff\left(\frac{1}{2}\right) = f(-6) = \frac{-24+1}{-18-2} = \frac{23}{20}$$

(c) $ff^{-1}(4) = 4$

or $f^{-1}(4) = \frac{2(4)+1}{3(4)-4} = \frac{9}{8}$

$$ff^{-1}(4) = f\left(\frac{9}{8}\right) = \frac{4\left(\frac{9}{8}\right)+1}{3\left(\frac{9}{8}\right)-2}$$

$$= \frac{\frac{11}{2}}{\frac{11}{8}}$$

$= 4$

17 $f(x) = \frac{2}{x-1}$

$g(x) = \lambda x^2 - 1$

(a) $f(3) = \frac{2}{3-1} = 1$

$g(1) = \lambda - 1$

$gf(3) = \frac{1}{5}$

$\Rightarrow \lambda - 1 = \frac{1}{5}$

$\lambda = 1\frac{1}{5}$

(b) $f^2(x) = ff(x)$

$$= f\left(\frac{2}{x-1}\right)$$

$$= \frac{2}{\frac{2}{x-1} - 1}$$

$$= \frac{2}{\frac{2-x+1}{x-1}}$$

$$= \frac{2x-2}{3-x},\ x \neq 3$$

$a = 2, b = -2$
$c = -1, d = 3$

18 $f(x) = 3x - 2, x \in \mathbb{R}$

$g(x) = \frac{2}{x-2}, x \in \mathbb{R}, x \neq 2$

(a) $y = 3x - 2$

$x = 3y - 2$

$y = \frac{x+2}{3}$

$f^{-1}(x) = \frac{x+2}{3}$

$y = \frac{2}{x-2}$

$x = \frac{2}{y-2}$

$xy - 2x = 2$

$y = \frac{2+2x}{x}$

$g^{-1}(x) = \frac{2+2x}{x}, x \neq 0$

(b) $fg(x) = x$

$f\left(\frac{2}{x-2}\right) = x$

$\Rightarrow \frac{6}{x-2} - 2 = x$

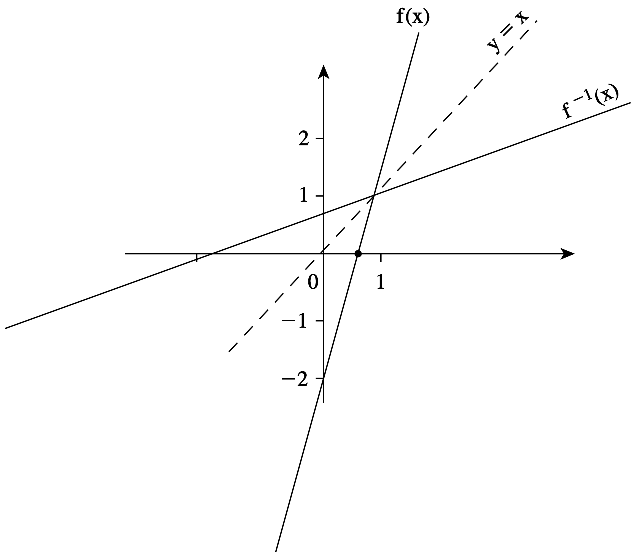
$\Rightarrow 6 - 2(x-2) = x(x-2)$

$\Rightarrow 6 - 2x + 4 = x^2 - 2x$

$x^2 = 10$

$x = \pm\sqrt{10}$

(c)



19 (a) $k = -3$

(b) $fg = f(2x + 1)$

$= \frac{3}{2x + 1 + 3}$

$= \frac{3}{2x + 4}$

$fg : x \rightarrow \frac{3}{2x + 4}, x \in \mathbb{R}, x \neq -2$

fg is not defined at $x = -2$

(c) $y = \frac{3}{x + 3}$

$x = \frac{3}{y + 3}$

$xy + 3x = 3$

$y = \frac{3 - 3x}{x}$

$f^{-1}(x) = \frac{3 - 3x}{x}, \ x \neq 0$

(d) $f^{-1}(a) = g(4)$

$\frac{3 - 3a}{a} = 2(4) + 1$

$3 - 3a = 9a$

$12a = 3$

$a = \frac{1}{4}$

20 (a) $f(x) = \frac{4x}{x - 1}, g(x) = \frac{x + \lambda}{x}$

$y = \frac{4x}{x - 1}$

$x = \dfrac{4y}{y-1}$

$xy - x = 4y$

$xy - 4y = x$

$y = \dfrac{x}{x-4}$

$f^{-1}(x) = \dfrac{x}{x-4}$

$f^{-1}: x \to \dfrac{x}{x-4}, x \in \mathbb{R}, x \neq 4.$

(b) $f^{-1}(5) = \dfrac{5}{5-4} = 5.$

$g(5) = 5.$

$\dfrac{5+\lambda}{5} = 5$

$\lambda = 20.$

21 (a) $3x^2 + 12x + 5 = a(x+b)^2 + c$

$= ax^2 + 2abx + ab^2 + c$

Coefficient of x^2: $a = 3$

Coefficient of x: $2ab = 12$

$b = 2$

Constants: $ab^2 + c = 5$

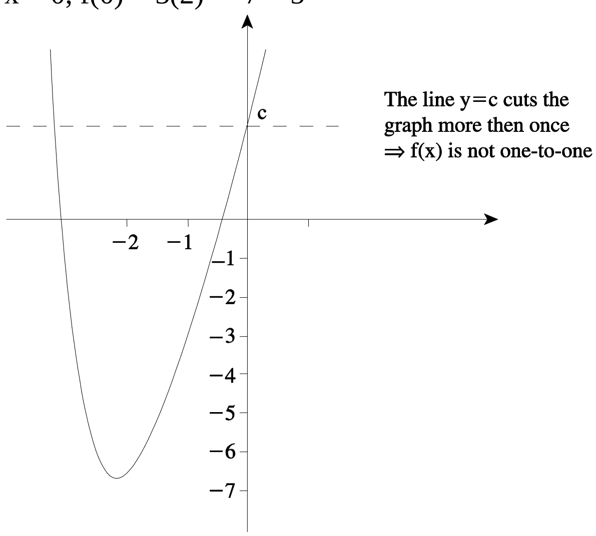
$12 + c = 5$

$c = -7$

$3x^2 + 12x + 5 = 3(x+2)^2 - 7$

(i) $y \geq -7$

(ii) $x = 0, f(0) = 3(2)^2 - 7 = 5$



The line y = c, c > -7, cuts the graph more than once ⇒ f(x) is not one-to-one.

(b) $g(x) = 3(x+2)^2 - 7, x \geq -2$

(i) $k = -2$

(ii) $y = 3(x+2)^2 - 7$

$x = 3(y + 2)^2 - 7$

$(y + 2)^2 = \frac{x + 7}{3}$

$y + 2 = \pm \sqrt{\frac{x + 7}{3}}$

$y = -2 \pm \sqrt{\frac{x + 7}{3}}$

$g^{-1}(x) = -2 + \sqrt{\frac{x + 7}{3}}$

22 (a) $f(1) = 4(1) - 1 = 3$

(b) $f(3{\cdot}5) = 2(3{\cdot}5) + 5 = 12$

(c) $f(5) = 5^2 + 3 = 28$

(d) $ff\left(\frac{1}{2}\right) = f\left(4\left(\frac{1}{2}\right) - 1\right) = f(1) = 4 - 1 = 3$

23 For $x < 0$; Gradient $= \frac{2}{+2} = +1$

Equation: $y - 2 = 1(x - 0)$

$y = x + 2$

For $x = 1$, $y = 2$

For $x > 1$, Gradient $= \frac{4 - 0}{4 - 1} = \frac{4}{3}$

$y - 4 = \frac{4}{3}(x - 4)$

$y = \frac{4}{3}x - \frac{16}{3} + 4$

$y = \frac{4}{3}x - \frac{4}{3}$

$$\therefore f(x) = \begin{cases} x + 2, x < 0 \\ 2, x = 1 \\ \frac{4}{3}x - \frac{4}{3}, x > 1 \end{cases}$$

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