Chapter 3 Principle of Mathematical Induction

Try these 3.1

(a) $\sum_{r=1}^{12} r^2 = \frac{12(12+1)(2(12)+1)}{6}$

$= \frac{(12)(13)(25)}{6}$

$= 650$

(b) $\sum_{r=10}^{30} r^2 = \sum_{r=1}^{30} r^2 - \sum_{r=1}^{9} r^2$

$= \frac{30(31)(61)}{6} - \frac{9(9+1)(18+1)}{6}$

$= 9455 - 285 = 9170$

Try these 3.2

(a) $\sum_{r=1}^{20} r(r+3)$

$= \sum_{r=1}^{20} r^2 + 3\sum_{r=1}^{20} r$

$= \frac{20(21)(41)}{6} + 3\left[\frac{20(21)}{2}\right] = 3500$

(b) $\sum_{r=10}^{25} 2r(r+1) = \sum_{r=1}^{25} 2r(r+1) - \sum_{r=1}^{9} 2r(r+1)$

$= 2\sum_{r=1}^{25} r^2 + 2\sum_{r=1}^{25} r - 2\sum_{r=1}^{9} r^2 - 2\sum_{r=1}^{9} r$

$= \frac{2(25)(26)(51)}{6} + \frac{2(25)(26)}{2} - \frac{2(9)(10)(19)}{6} - \frac{2(9)(10)}{2}$

$= 11\,040$

(c) $\sum_{r=1}^{n} r(r^2+2r) = \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2$

$= \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6}$

$= \frac{n(n+1)}{12}[3n(n+1) + 4(2n+1)]$

$= \frac{n(n+1)}{12}[3n^2 + 11n + 4] = \frac{1}{12} n(n+1)(3n^2 + 11n + 4)$

Exercise 3A

1 $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{343}, \ldots$

$u_1 = \frac{1}{3^1}, u_2 = \frac{1}{3^2}, u_3 = \frac{1}{3^3} \ldots$

$u_n = \frac{1}{3^n}$

2 16, 13, 10, 7, 4, …

Sequence decreasing by 3

$u_n = an + b$

$u_n = -3n + b$

$u_1 = 16 \Rightarrow 16 = -3 + b$

$b = 19$

$u_n = -3n + 19$

3 $\frac{1}{2 \times 5}, \frac{1}{3 \times 7}, \frac{1}{4 \times 9}, \frac{1}{5 \times 11}, \ldots$

2, 3, 4, ….

n + 1

5, 7, 9, …

2n + 3

$\therefore u_n = \frac{1}{(n+1)(2n+3)}$

4 8 + 16 + 32 + 64 + 128 + 256 + 512

$u = 2^{r+2}$

$\sum_{r=1}^{7} 2^{r+2}$

5 9 + 12 + 15 + … + 30

$u_r = 3r + 6$

$\sum_{r=1}^{8} (3r + 6)$

6 $4 \times 5 + 5 \times 6 + 6 \times 7 + \ldots + 10 \times 11$

$u_r = (r + 3)(r + 4)$

$\sum_{r=1}^{7} (r+3)(r+4)$

7 $\sum_{r=1}^{n} (6r - 5)$

$u_n = 6n - 5$

8 $\sum_{r=1}^{n} (4r^2 - 3)$

$u_n = 4n^2 - 3$

9 $\sum_{r=1}^{2n} (r^3 + r^2)$

$u_n = n^3 + n^2$

10 $\sum_{r=1}^{4n} (6r^3 + 2)$

$u_n = 6n^3 + 2$

11 $\sum_{r=1}^{n+2} 3^{2r-1}$

$u_n = 3^{2n-1}$

12 $u_{16} = 7(16) + 3 = 115$

13 $u_8 = 3(9)^2 - 1 = 242$

14 $r = 14,\ u_{10} = \dfrac{1}{4(14) - 2} = \dfrac{1}{54}$

15 $\sum_{r=1}^{25}(r - 2) = \sum_{r=1}^{25} r - \sum_{r=1}^{25} 2$

$= \dfrac{25(26)}{2} - 25(2)$

$= 275$

16 $\sum_{r=1}^{30}(6r + 3) = 6\sum_{r=1}^{30} r + \sum_{r=1}^{30} 3$

$= 6\dfrac{(30)(31)}{2} + 30(3)$

$= 2880$

17 $\sum_{r=1}^{50} r(r + 2)$

$= \sum_{r=1}^{50} r^2 + 2\sum_{r=1}^{50} r$

$= \dfrac{50(51)(101)}{6} + \dfrac{2(50)(51)}{2}$

$= 45\,475$

18 $\sum_{r=1}^{10} r^2(r + 4)$

$= \sum_{r=1}^{10} r^3 + 4\sum_{r=1}^{10} r^2$

$= \dfrac{(10)^2(11)^2}{4} + \dfrac{4(10)(11)(21)}{6}$

$= 3025 + 1540$

$= 4565$

19 $\sum_{r=1}^{45} 6r(r + 1)$

$= 6\sum_{r=1}^{45} r^2 + 6\sum_{r=1}^{45} r$

$= \dfrac{6(45)(46)(91)}{6} + \dfrac{6(45)(46)}{2}$

$= 194\,580$

20 $\sum_{r=5}^{12}(r + 4) = \sum_{r=1}^{12}(r + 4) - \sum_{r=1}^{4}(r + 4)$

$= \sum_{r=1}^{12} r + \sum_{r=1}^{12} 4 - \sum_{r=1}^{4} r - \sum_{r=1}^{4} 4$

$= \dfrac{12(13)}{2} + (4)(12) - \dfrac{(4)(5)}{2} - (4)(4)$

$= 78 + 48 - 10 - 16$

$= 100$

21 $\sum_{r=10}^{25}(r^2 - 3)$

$= \sum_{r=1}^{25}(r^2-3) - \sum_{r=1}^{9}(r^2-3)$

$= \sum_{r=1}^{25} r^2 - \sum_{r=1}^{25} 3 - \sum_{r=1}^{9} r^2 + \sum_{r=1}^{9} 3$

$= \frac{(25)(26)(51)}{6} - (25)(3) - \frac{9(10)(19)}{6} + (9)(3)$

= 5192

22 $\sum_{r=15}^{30} r(3r-2)$

$= 3\sum_{r=1}^{30} r^2 - 2\sum_{r=1}^{30} r - 3\sum_{r=1}^{14} r^2 + 2\sum_{r=1}^{14} r$

$= \frac{3(30)(31)(61)}{6} - \frac{2(30)(31)}{2} - \frac{3(14)(15)(29)}{6} + \frac{2(14)(15)}{2}$

= 24 600

23 $\sum_{r=9}^{40}(2r+1)(5r+2)$

$= \sum_{r=1}^{40}(10r^2+9r+2) - \sum_{r=1}^{8}(10r^2+9r+2).$

$= 10\sum_{r=1}^{40} r^2 + 9\sum_{r=1}^{40} r + \sum_{r=1}^{40} 2 - 10\sum_{r=1}^{8} r^2 - 9\sum_{r=1}^{8} r - \sum_{r=1}^{8} 2$

$= \frac{10(40)(41)(81)}{6} + \frac{9(40)(41)}{2} + (40)(2) - \frac{10(8)(9)(17)}{6} - \frac{9(8)(9)}{2} - (8)(2)$

= 226 480

24 $\sum_{r=1}^{n}(r+4)$

$= \sum_{r=1}^{n} r + \sum_{r=1}^{n} 4$

$= \frac{n(n+1)}{2} + 4n$

$= \frac{n}{2}(n+1+8)$

$= \frac{1}{2} n(n+9)$

25 $\sum_{r=1}^{n} 3r(r+1)$

$= \sum_{r=1}^{n}(3r^2+3r)$

$= 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$

$= \frac{3n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$

$= \frac{n(n+1)}{2}[2n+1+3]$

$= \frac{n(n+1)}{2}(2n+4)$

$= n(n+1)(n+2)$

26 $\sum_{r=1}^{n} 4r(r-1)$

$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r$

$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2}$

$= \frac{n(n+1)}{3}[2(2n+1)-6]$

$= \frac{n(n+1)}{3}(4n-4)$

$= \frac{4n(n+1)(n-1)}{3}$

27 $\sum_{r=1}^{n} r^2(r+3)$

$= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2$

$= \frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6}$

$= \frac{n(n+1)}{4}[n(n+1)+2(2n+1)]$

$= \frac{n(n+1)}{4}[n^2+5n+2]$

28 $\sum_{r=n+1}^{2n} 2r(r-1)$

$= \sum_{r=1}^{2n}(2r^2-2r) - \sum_{r=1}^{n}(2r^2-2r)$

$= 2\sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{2n} r - 2\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{2(2n)(2n+1)(4n+1)}{6} - \frac{2(2n)(2n+1)}{2} - \frac{2n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$= n\left[\frac{2}{3}(2n+1)(4n+1) - 2(2n+1) - \frac{1}{3}(n+1)(2n+1) + (n+1)\right]$

$= \frac{n}{3}\left[2(8n^2+6n+1) - (12n+6) - (2n^2+3n+1) + 3n+3\right]$

$= \frac{n}{3}(14n^2-2) = \frac{2n(7n^2-1)}{3}$

29 $\sum_{r=n+1}^{2n} r(r+4)$

$= \sum_{r=1}^{2n}(r^2+4r) - \sum_{r=1}^{n}(r^2+4r)$

$$= \sum_{r=1}^{2n} r^2 + 4\sum_{r=1}^{2n} r - \sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r$$

$$= \frac{2n\,(2n+1)\,(4n+1)}{6} + \frac{4\,(2n)\,(2n+1)}{2} - \frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2}$$

$$= \frac{n}{6}[2(8n^2+6n+1) + 24\,(2n+1) - (2n^2+3n+1) - 12\,(n+1)]$$

$$= \frac{n}{6}[14n^2+45n+13]$$

30 $$\sum_{r=n+1}^{2n} (r+1)\,(r-1)$$

$$= \sum_{r=1}^{2n} (r^2-1) - \sum_{r=1}^{n} (r^2-1)$$

$$= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{2n} 1 - \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 1$$

$$= \frac{2n(2n+1)\,(4n+1)}{6} - 2n - \frac{n(n+1)\,(2n+1)}{6} + n$$

$$= \frac{2n(8n^2+6n+1)}{6} - \frac{n(2n^2+3n+1)}{6} - n$$

$$= \frac{n}{6}\,[16n^2+12n+2-2n^2-3n-1-6]$$

$$= \frac{n}{6}(14n^2+9n-5)$$

$$= \frac{n}{6}(14n-5)(n+1)$$

Exercise 3B

1 RTP $\sum_{r=1}^{n} (3r-2) = \frac{1}{2}n\,(3n-1)$

Proof: When n = 1, LHS = $\sum_{r=1}^{1} (3r-2) = 3(1) - 2 = 1$, RHS $= \frac{1}{2}(1)\,(3(1)-1) = \frac{1}{2}\times 2 = 1$

∴ LHS = RHS

Hence when n=1, $\sum_{r=1}^{n} (3r-2) = \frac{1}{2}\,n(3n-1)$

Assume that the statement is true for n = k

i.e. $\sum_{r=1}^{k} (3r-2) = \frac{1}{2}\,k\,(3k-1)$

RTP the statement true for n = k + 1

i.e. $\sum_{r=1}^{k+1} (3r-2) = \frac{1}{2}(k+1)\,(3(k+1)-1)$

Proof: $\sum_{r=1}^{k+1} (3r-2) = \sum_{r=1}^{k} (3r-2) + 3(k+1) - 2$

$$= \frac{1}{2}k\,(3k-1) + (3k+1)$$

$=\frac{1}{2}[3k^2 - k + 6k + 2]$

$=\frac{1}{2}(3k^2 + 5k + 2)$

$=\frac{1}{2}(k+1)(3k+2)$

$=\frac{1}{2}(k+1)(3(k+1)-1)$

Hence $\sum_{r=1}^{k+1}(3r-2) = \frac{1}{2}(k+1)(3(k+1)-1)$

$\therefore$ by PMI $\sum_{r=1}^{n}(3r-2) = \frac{1}{2}n(3n-1)$

2 RTP $\sum_{r=1}^{n}(4r-3) = n(2n-1)$

Proof:

When n = 1, $\sum_{r=1}^{1}(4r-3) = 4(1)-3 = 4-3 = 1$

RHS = 1(2(1)–1) = 2–1=1

$\Rightarrow$ LHS = RHS

Hence when n = 1, $\sum_{r=1}^{n}(4r-3) = n(2n-1)$

Assume true for n = k i.e. $\sum_{r=1}^{k}(4r-3) = k(2k-1)$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1}(4r-3) = (k+1)(2(k+1)-1)$

Proof: $\sum_{r=1}^{k+1}(4r-3) = \sum_{r=1}^{k}(4r-3) + 4(k+1) - 3$

= k (2k – 1) + 4k + 1

= $2k^2$ – k + 4k + 1

= $2k^2$ + 3k + 1

= (2k + 1) (k + 1)

= (k + 1) (2(k + 1) – 1)

$\therefore$ $\sum_{r=1}^{k+1}(4r-3) = (k+1)(2(k+1)-1)$

Hence by PMI $\sum_{r=1}^{n}(4r-3) = n(2n-1)$

3 RTP $\sum_{r=1}^{n}(2r-1)(2r) = \frac{1}{3}n(n+1)(4n-1)$

Proof:

When n = 1, LHS = $\sum_{r=1}^{1}(2r-1)(2r) = (2(1)-1)(2(1)) = 2$

RHS = $\frac{1}{3}(1)(1+1)(4(1)-1) = \frac{1}{\cancel{3}} \times 2 \times \cancel{3} = 2$

$\therefore$ LHS = RHS

Assume true for n = k, i.e. $\sum_{r=1}^{k}(2r-1)(2r)=\frac{1}{3}k(k+1)(4k-1)$

RTP true for n = k + 1, i.e. $\sum_{r=1}^{k+1}(2r-1)(2r)=\frac{1}{3}(k+1)(k+1+1)(4(k+1)-1)$

Proof: $\sum_{r=1}^{k+1}(2r-1)(2r)=\sum_{r=1}^{k}(2r-1)(2r)+(2(k+1)-1)(2(k+1))$

$=\frac{1}{3}k(k+1)(4k-1)+(2k+1)(2k+2)$

$=\frac{1}{3}(k+1)[k(4k-1)+6(2k+1)]$

$=\frac{1}{3}(k+1)[4k^2-k+12k+6]$

$=\frac{1}{3}(k+1)(4k^2+11k+6)$

$=\frac{1}{3}(k+1)(k+2)(4k+3)$

$=\frac{1}{3}(k+1)(k+1+1)(4(k+1)-1)$

Hence $\sum_{r=1}^{k+1}(2r-1)(2r)=\frac{1}{3}(k+1)(k+1+1)(4(k+1)-1)$

Hence by PMI $\sum_{r=1}^{n}(2r-1)(2r)=\frac{1}{3}n(n+1)(4n-1)$

4 RTP $\sum_{r=1}^{n}(r^2+r^3)=\frac{n(n+1)(n+2)(3n+1)}{12}$

Proof:

When n = 1, LHS $=\sum_{r=1}^{1}(r^2+r^3)=1^2+1^3=2$

RHS $=\frac{(1)(1+1)(1+2)(3(1)+1)}{12}=\frac{2\times \cancel{3\times 4}}{\cancel{12}}=2$

$\therefore$ LHS = RHS

When n = 1, $\sum_{r=1}^{n}(r^2+r^3)=\frac{n(n+1)(n+2)(3n+1)}{12}$

Assume true for n = k i.e. $\sum_{r=1}^{k}(r^2+r^3)=\frac{k(k+1)(k+2)(3k+1)}{12}$

RTP true for n = k+1 i.e. $\sum_{r=1}^{k+1}(r^2+r^3)=\frac{(k+1)(k+1+1)(k+1+2)(3(k+1)+1)}{12}$

Proof: $\sum_{r=1}^{k+1}(r^2+r^3)=\sum_{r=1}^{k}(r^2+r^3)+(k+1)^2+(k+1)^3$

$=\frac{k(k+1)(k+2)(3k+1)}{12}+(k+1)^2+(k+1)^3$

$=\frac{k+1}{12}[k(k+2)(3k+1)+12(k+1)+12(k+1)^2]$

$=\frac{1}{12}(k+1)[3k^3+7k^2+2k+12k+12+12k^2+24k+12]$

$$=\frac{1}{12}(k+1)(3k^3+19k^2+38k+24)$$

$$=\frac{1}{12}(k+1)(k+2)(3k^2+13k+12)$$

$$=\frac{1}{12}(k+1)(k+2)(k+3)(3k+4)$$

$$=\frac{1}{12}(k+1)((k+1)+1)((k+1)+2)(3(k+1)+1)$$

Hence by PMI $\sum_{r=1}^{n}(r^2+r^3)=\frac{n(n+1)(n+2)(3n+1)}{12}$

5 RTP $\sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4}$

Proof: when n = 1, LHS = $\sum_{r=1}^{1} r^3=1^3=1$

RHS $=\frac{(1)^2(1+1)^2}{4}=\frac{4}{4}=1$

$\therefore$ LHS = RHS

When n = 1, $\sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4}$

Assume true for n = k i.e. $\sum_{r=1}^{k} r^3=\frac{k^2(k+1)^2}{4}$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1} r^3=\frac{(k+1)^2((k+1)+1)^2}{4}$

Proof: $\sum_{r=1}^{k+1} r^3=\sum_{r=1}^{k} r^3+(k+1)^3$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{(k+1)^2}{4}[k^2+4(k+1)]$$

$$=\frac{1}{4}(k+1)^2(k^2+4k+4)$$

$$=\frac{1}{4}(k+1)^2(k+2)^2$$

$\therefore$ by PMI $\sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4}$

6 RTP $\sum_{r=1}^{n}\frac{1}{r(r+1)}=\frac{n}{n+1}$

Proof: When n=1, LHS = $\sum_{r=1}^{1}\frac{1}{r(r+1)}=\frac{1}{1(1+1)}=\frac{1}{2}$

RHS = $\frac{1}{1+1}=\frac{1}{2}$

$\therefore$ LHS = RHS

Hence when n=1, $\sum_{r=1}^{n}\frac{1}{r(r+1)}=\frac{n}{n+1}$

Assume true for n = k i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{(k+1)+1}$

Proof: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right]$

$= \frac{1}{k+1}\left[\frac{k(k+2)+1}{k+2}\right]$

$= \frac{1}{k+1}\left[\frac{k^2+2k+1}{k+2}\right]$

$= \frac{1}{\cancel{k+1}} \frac{(k+1)^{\cancel{2}}}{k+2}$

$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$

Hence by PMI $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{n}{n+1}$

7 RTP $\sum_{r=1}^{n} (-1)^{r+1} r^2 = \frac{(-1)^{n+1}(n)(n+1)}{2}$

Proof: when n = 1, LHS = $\sum_{r=1}^{1} (-1)^{r+1} r^2 = (-1)^2 (1)^2 = 1$

RHS $= \frac{(-1)^2 (1)(2)}{2} = 1$

LHS = RHS

Hence when n = 1, $\sum_{r=1}^{n} (-1)^{r+1} r^2 = \frac{(-1)^{n+1} n(n+1)}{2}$

Assume true for n = k, i.e. $\sum_{r=1}^{k} (-1)^{r+1} r^2 = \frac{(-1)^{k+1}(k)(k+1)}{2}$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1} (-1)^{r+1} r^2 = \frac{(-1)^{k+2}(k+1)(k+1+1)}{2}$

Proof: $\sum_{r=1}^{k+1} (-1)^{r+1} r^2 = \sum_{r=1}^{k} (-1)^{r+1} r^2 + (-1)^{k+2}(k+1)^2$

$= \frac{(-1)^{k+1}(k)(k+1)}{2} + (-1)^{k+2}(k+1)^2$

$= \frac{(-1)^{k+1}(k+1)}{2}[k + (-1)^1 2(k+1)]$

$= \frac{(-1)^{k+1}(k+1)}{2}[-k-2]$

$= \frac{(-1)^{k+1}(k+1)(-1)^1(k+2)}{2}$

$= \frac{(-1)^{k+2}(k+1)(k+2)}{2}$

Hence by PMI $\sum_{r=1}^{n}(-1)^{r+1}r^2 = \frac{(-1)^{n+1}(n)(n+1)}{2}$

8 RTP $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Proof: when n = 1, LHS $= \sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)} = \frac{1}{1(1+1)(1+2)} = \frac{1}{2\times 3} = \frac{1}{6}$

RHS $= \frac{(1)(1+3)}{4(1+1)(1+2)} = \frac{\cancel{4}}{\cancel{4}\times 2\times 3} = \frac{1}{6}$

$\therefore$ LHS = RHS

When n = 1, $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Assume true for n = k i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$

RTP true for n = k + 1, i.e. $\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \frac{(k+1)((k+1)+3)}{4((k+1)+1)((k+1)+2)}$

Proof: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{1}{4(k+1)(k+2)}\left[k(k+3) + \frac{4}{k+3}\right]$

$= \frac{1}{4(k+1)(k+2)}\left[\frac{k(k+3)(k+3)+4}{k+3}\right]$

$= \frac{1}{4(k+1)(k+2)}\left[\frac{k^3+6k^2+9k+4}{k+3}\right]$

$= \frac{1}{4(k+1)(k+2)} \frac{(k+1)(k^2+5k+4)}{k+3}$

$= \frac{(k+1)(k+4)\cancel{(k+1)}}{4\cancel{(k+1)}(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \frac{(k+1)((k+1)+3)}{4((k+1)+1)((k+1)+2)}$

Hence by PMI $\sum_{r=1} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

9 RTP $\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{6n+4}$

Proof: when n = 1, LHS $= \sum_{r=1}^{1} \frac{1}{(3r-1)(3r+2)} = \frac{1}{(3-1)(3+2)} = \frac{1}{10}$

$\text{RHS} = \frac{1}{6(1)+4} = \frac{1}{10}$

$\Rightarrow$ LHS = RHS

When n = 1, $\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{6n+4}$

Assume true for n = k i.e. $\sum_{r=1}^{k} \frac{1}{(3r-1)(3r+2)} = \frac{k}{6k+4}$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1} \frac{1}{(3r-1)(3r+2)} = \frac{k+1}{6(k+1)+4}$

Proof: $\sum_{r=1}^{k+1} \frac{1}{(3r-1)(3r+2)} = \sum_{r=1}^{k} \frac{1}{(3r-1)(3r+2)} + \frac{1}{(3(k+1)-1)(3(k+1)+2)}$

$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$

$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$

$= \frac{1}{2(3k+2)}\left[k + \frac{2}{3k+5}\right]$

$= \frac{1}{2(3k+2)}\left[\frac{k(3k+5)+2}{3k+5}\right]$

$= \frac{1}{2(3k+2)}\left[\frac{3k^2+5k+2}{3k+5}\right]$

$= \frac{\cancel{(3k+2)}(k+1)}{2(3k+5)\cancel{(3k+2)}}$

$= \frac{k+1}{6k+10} = \frac{k+1}{6(k+1)+4}$

Hence by PMI $\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{6n+4}$

10 RTP $3^{4n} - 1 = 16A$, $A \in \mathbb{Z}$, $n \geq 1$
Proof: when n = 1, LHS $= 3^{4(1)} - 1 = 3^4 - 1 = 81 - 1 = 80$
$= 16(5)$
$\therefore$ when n = 1, $3^{4n} - 1$ is divisible by 16
Assume true for n = k, i.e. $3^{4k} - 1 = 16A$
RTP true for n = k + 1, i.e. $3^{4(k+1)} - 1 = 16B$
Proof: $3^{4k+4} - 1$
$= 3^{4k+4} + 16A - 3^{4k}$
$= 3^{4k} \times 3^4 - 3^{4k} + 16A$
$= 3^{4k}(3^4 - 1) + 16A$
$= 3^{4k}(80) + 16A$
$= 16[5(3^{4k}) + A]$
$= 16B$, $B = 5(3^{4k}) + A \in \mathbb{Z}$
Hence by PMI $3^{4n} - 1$ is divisible by 16

11 RTP $n^4 + 3n^2 = 4A$, $A \in \mathbb{Z}$, $n \geq 1$
Proof: when n = 1, LHS $= 1^4 + 3(1)^2$
$= 1 + 3 = 4$
$= 4(1)$

∴ when n = 1, $n^4 + 3n^2$ is divisible by 4
Assume true for n = k, i.e. $k^4 + 3k^2 = 4A$
RTP true for n = k + 1, i.e. $(k + 1)^4 + 3(k+1)^2 = 4B$
Proof:
$(k+1)^4 + 3(k+1)^2$
$= k^4 + 4k^3 + 6k^2 + 4k + 1 + 3k^2 + 6k + 3$
$= (k^4 + 3k^2) + 4k^3 + 4k + 4 + 6k^2 + 6k$
$= 4A + 4k^3 + 4k + 4 + 6k(k + 1)$
$= 4A + 4k^3 + 4k + 4 + 6(2c)$ Since k(k + 1) is the product of two consecutive, integers then k(k + 1) is divisible by 2. i.e. k(k + 1) = 2c
$= 4[A + k^3 + k + 1 + 3c]$
$= 4B$
Hence by PMI $n^4 + 3n^2$ is divisible by 4

Review exercise 3

1 $6 \times 7 + 8 \times 10 + 10 \times 13 + \ldots$

(a) $u_n = (2n + 4)(3n + 4)$

(b) $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (2r + 4)(3r + 4)$

$= \sum_{r=1}^{n} (6r^2 + 20r + 16)$

$= 6\sum_{r=1}^{n} r^2 + 20\sum_{r=1}^{n} r + \sum_{r=1}^{n} 16$

$= \frac{6n(n+1)(2n+1)}{6} + \frac{20n(n+1)}{2} + 16n$

$= n(n + 1)(2n + 1) + 10n(n + 1) + 16n$

$= n[2n^2 + 3n + 1 + 10n + 10 + 16]$

$= n(2n^2 + 13n + 27)$

2 (a) $\sum_{r=1}^{n} r(3r - 2)$

$= 3\sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r$

$= \frac{3n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2}$

$= \frac{n(n+1)}{2}[2n + 1 - 2]$

$= \frac{n(n+1)(2n-1)}{2}$

(b) (i) $\sum_{r=1}^{20} r(3r - 2) = \frac{20(21)(39)}{2} = 8190$

(ii) $\sum_{r=1}^{100} r(3r - 2) = \sum_{r=1}^{100} r(3r - 2) - \sum_{r=1}^{20} r(3r - 2)$

$= \frac{(100)(101)(199)}{2} - 8190$

$= 996\,760$

3 RTP $\sum_{r=1}^{n} 2r(r-5) = \frac{2n(n+1)(n-7)}{3}$

Proof:

When n=1, LHS $= \sum_{r=1}^{1} 2r(r-5)$

$= 2(1-5) = -8$

RHS $= \frac{2(1)(1+1)(1-7)}{3} = \frac{2 \times 2 \times (-6)}{3} = -8$

$\therefore$ when n=1, LHS = RHS

$\Rightarrow \sum_{r=1}^{n} 2r(r-5) = \frac{2n(n+1)(n-7)}{3}$ when $n = 1$

Assume true for n = k, i.e. $\sum_{r=1}^{k} 2r(r-5) = \frac{2k(k+1)(k-7)}{3}$

RTP true for n = k + 1, i.e. $\sum_{r=1}^{k+1} 2r(r-5) = \frac{2(k+1)(k+1+1)(k+1-7)}{3}$

Proof:

$\sum_{r=1}^{k+1} 2r(r-5) = \sum_{r=1}^{k} 2r(r-5) + 2(k+1)(k+1-5)$

$= \frac{2k(k+1)(k-7)}{3} + 2(k+1)(k-4)$

$= \frac{2(k+1)}{3}[k(k-7)+3(k-4)]$

$= \frac{2(k+1)}{3}[k^2 - 4k - 12]$

$= \frac{2(k+1)}{3}(k+2)(k-6)$

$= \frac{2(k+1)(k+1+1)(k+1-7)}{3}$

Hence by PMI $\sum_{r=1}^{n} 2r(r-5) = \frac{2n(n+1)(n-7)}{3}$

4 $a_n = 3^{2n-1} + 1$

$a_{n+1} = 3^{2(n+1)-1} + 1 = 3^{2n+1} + 1$

$a_{n+1} - a_n = 3^{2n+1} + 1 - 3^{2n-1} - 1$

$= 3^{2n+1} - 3^{2n-1}$

$= 3^{2n-1}[3^2 - 1]$

$= 8(3^{2n-1})$

RTP $a_n = 3^{2n-1} + 1 = 4A$, $A \in Z$ for all $n \geq 1$

Proof: when n=1, $3^{2-1} + 1 = 3 + 1 = 4(1)$

$\therefore$ when n=1, a_n is divisible by 4

Assume true for n = k i.e. $a_k = 4A$

RTP true for n = k + 1 i.e. $a_{k+1} = 4B$,

Proof: From above

$a_{n+1} - a_n = 8(3^{2n-1})$

$\Rightarrow a_{k+1} - a_k = 8(3^{2k-1})$

$\Rightarrow a_{k+1} - 4A = 8(3^{2k-1})$

$a_{k+1} = 4A + 8(3^{2k-1})$

$= 4[A + 2(3^{2k-1})]$

= 4B
∴ a_{k+1} is divisible by 4
Hence by PMI a_n is divisible by 4

5 $\sum_{r=1}^{n} 2r(r^2 - 1)$

$= 2\sum_{r=1}^{n} r^3 - 2\sum_{r=1}^{n} r$

$= \frac{2n^2(n+1)^2}{4} - \frac{2n(n+1)}{2}$

$= \frac{n(n+1)}{2}[n(n+1) - 2]$

$= \frac{n(n+1)}{2}[n^2 + n - 2]$

$= \frac{n(n+1)(n+2)(n-1)}{2}$

Proof by induction:

RTP $\sum_{r=1}^{n} 2r(r^2 - 1) = \frac{n(n+1)(n+2)(n-1)}{2}$

Proof: when n=1, LHS = $2(1^2 - 1) = 0$

RHS $= \frac{(1)(2)(3)(0)}{2} = 0$

∴ LHS = RHS

Hence when n = 1, $\sum_{r=1}^{n} 2r(r^2 - 1) = \frac{n(n+1)(n-1)(n+2)}{2}$

Assume true for n = k i.e. $\sum_{r=1}^{k} 2r(r^2 - 1) = \frac{k(k+1)(k-1)(k+2)}{2}$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1} 2r(r^2 - 1) = \frac{(k+1)(k+1+1)(k+1-1)(k+1+2)}{2}$

Proof: $\sum_{r=1}^{k+1} 2r(r^2 - 1)$

$= \sum_{r=1}^{k} 2r(r^2 - 1) + 2(k+1)((k+1)^2 - 1)$

$= \frac{k(k+1)(k-1)(k+2)}{2} + 2(k+1)(k^2 + 2k)$

$= \frac{k(k+1)}{2}[(k-1)(k+2) + 4(k+2)]$

$= \frac{k(k+1)(k+2)}{2}(k - 1 + 4)$

$= \frac{k(k+1)(k+2)(k+3)}{2} = \frac{(k+1)(k+1+1)(k+1-1)(k+1+2)}{2}$

Hence by PMI $\sum_{r=1}^{k} 2r(r^2 - 1) = \frac{n(n+1)(n-1)(n+2)}{2}$

6 $a_n = 5^{2n+1} + 1$
$a_{n+1} = 5^{2(n+1)+1} + 1$
$= 5^{2n+3} + 1$
$a_{n+1} - a_n = 5^{2n+3} + 1 - 5^{2n+1} - 1$

$= 5^{2n+3} - 5^{2n+1}$
$= 5^{2n+1}(5^2 - 1)$
$= (24)(5^{2n+1})$
RTP: $a_n = 6A, \forall n \geq 0$
Proof: when $n = 0$, $a_0 = 5^1 + 1 = 6 = 6(1)$
Hence when $n = 0$, a_n is divisible by 6
Assume true for $n = k$ i.e. $a_k = 6A$
RTP true for $n = k + 1$, i.e. $a_{k+1} = 6B$
Proof: $a_{k+1} - a_k = 24(5^{2k+1})$, from above
$\Rightarrow a_{k+1} - 6A = 6(4)(5^{2k+1})$
$a_{k+1} = 6A + 6(4)5^{2k+1}$
$= 6[A + 4(5^{2k+1})]$
$= 6B$
Hence by PMI a_n is divisible by 6. $\forall n \geq 0$

7 $\displaystyle\sum_{r=1}^{n}(6r^3 + 2)$

$$= 6\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} 2$$

$$= \frac{6n^2(n+1)^2}{4} + 2n$$

$$= \frac{n}{2}[3n(n+1)^2 + 4]$$

$$= \frac{n}{2}(3n^3 + 6n^2 + 3n + 4)$$

RTP $\displaystyle\sum_{r=1}^{n}(6r^3 + 2) = \frac{n}{2}(3n^3 + 6n^2 + 3n + 4)$

Proof: $n = 1$, LHS $= 6(1)^3 + 2 = 8$

RHS $= \frac{1}{2}(3 + 6 + 3 + 4) = \frac{16}{2} = 8$

$\therefore$ LHS = RHS

Hence when $n = 1$, $\displaystyle\sum_{r=1}^{n}(6r^3 + 2) = \frac{n}{3}(3n^3 + 6n^2 + 3n + 4)$

Assume true for $n = k$, i.e. $\displaystyle\sum_{r=1}^{k}(6r^3 + 2) = \frac{k}{2}(3k^3 + 6k^2 + 3k + 4)$

RTP true for $n = k + 1$ i.e. $\displaystyle\sum_{r=1}^{k+1}(6r^3 + 2) = \frac{k+1}{2}(3(k+1)^3 + 6(k+1)^2 + 3(k+1) + 4)$

Proof:

$$\sum_{r=1}^{k+1}(6r^3 + 2) = \sum_{r=1}^{k}(6r^3 + 2) + 6(k+1)^3 + 2$$

$$= \frac{k}{2}(3k^3 + 6k^2 + 3k + 4) + 6(k+1)^3 + 2$$

$$= \frac{1}{2}[3k^4 + 6k^3 + 3k^2 + 4k + 12(k^3 + 3k^2 + 3k + 1) + 4]$$

$$= \frac{1}{2}[3k^4 + 18k^3 + 39k^2 + 40k + 16]$$

$$= \frac{1}{2}(k+1)(3k^3 + 15k^2 + 24k + 16)$$

$= \frac{1}{2}(k+1)[3(k+1)^3 + 6(k+1)^2 + 3(k+1) + 4]$

Hence by PMI $\sum_{r=1}^{n}(6r^3 + 2) = \frac{n}{2}(3n^3 + 6n^2 + 3n + 4)$

8 RTP $\sum_{r=1}^{n}(r + 4) = \frac{1}{2}n(n+9)$

Proof:
when n = 1, L.H.S = 1 + 4 = 5

R.H.S = $\frac{1}{2}(1)(1+9) = \frac{10}{2} = 5$

$\therefore$ LHS = RHS

when n = 1, $\sum_{r=1}^{n}(r + 4) = \frac{1}{2}n(n+9)$

Assume true for n = k, i.e. $\sum_{r=1}^{k}(r + 4) = \frac{1}{2}k(k+9)$

RTP true for n = k+1, i.e. $\sum_{r=1}^{k+1}(r + 4) = \frac{1}{2}(k+1)(k+1+9)$

Proof: $\sum_{r=1}^{k+1}(r + 4) = \sum_{r=1}^{k}(r + 4) + (k+1+4)$

$= \frac{1}{2}k(k+9) + (k+5)$

$= \frac{1}{2}[k^2 + 9k + 2k + 10]$

$= \frac{1}{2}[k^2 + 11k + 10]$

$= \frac{1}{2}(k+1)(k+10)$

$= \frac{1}{2}(k+1)(k+1+9)$

Hence by PMI $\sum_{r=1}^{n}(r + 4) = \frac{1}{2}n(n+9)$

9 RTP $\sum_{r=1}^{n}4r(r-1) = \frac{4n(n+1)(n-1)}{3}$

Proof:

When n = 1, LHS = 4(1)(1 – 1) = 0

RHS = $\frac{4(1)(1+1)(1-1)}{3} = \frac{4\times 2\times 0}{3} = 0$

$\therefore$ LHS = RHS

$\therefore \sum_{r=1}^{n}4r(r-1) = \frac{4n(n+1)(n-1)}{3}$

Assume true for n = k i.e. $\sum_{r=1}^{k}4r(r-1) = \frac{4k(k+1)(k-1)}{3}$

RTP true for n = k + 1 i.e. $\sum_{r=1}^{k+1}4r(r-1) = \frac{4(k+1)(k+1+1)(k+1-1)}{3}$

Proof:

$$\sum_{r=1}^{k+1} 4r(r-1) = \sum_{r=1}^{k} 4r(r-1) + 4(k+1)(k+1-1)$$

$$= \frac{4k(k+1)(k-1)}{3} + 4(k+1)(k)$$

$$= \frac{4}{3}(k+1)\,k\,[k-1+3]$$

$$= \frac{4}{3}(k+1)(k)(k+2)$$

$$= \frac{4}{3}\;(k+1)(k+1+1)(k+1-1)$$

Hence by PMI $\sum_{r=1}^{n} 4r(r-1) = \frac{4}{3}n(n+1)(n-1)$

10 RTP $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{(n+1)}$

Proof:

when n = 1, LHS $= \frac{1}{1(1+1)} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

L.H.S = RHS

$\therefore$ when n = 1, $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{(n+1)}$

Assume true for n = k, i.e. $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{(k+1)}$

RTP true for n = k + 1, i.e. $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+1+1}$

Proof: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Hence by PMI $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

11 RTP $\sum_{r=1}^{n} 3(2^{r-1}) = 3(2^n - 1)$

Proof:

when n = 1, LHS = $3(2^{1-1}) = 3$

RHS = $3(2^1 - 1) = 3$

LHS = RHS

when n = 1, $\sum_{r=1}^{n} 3(2^{r-1}) = 3(2^n - 1)$

Assume true for n = k, i.e. $\sum_{r=1}^{k} 3(2^{r-1}) = 3(2^k - 1)$

RTP true for n = k + 1, i.e. $\sum_{r=1}^{k+1} 3(2^{r-1}) = 3(2^{k+1} - 1)$

Proof: $\sum_{r=1}^{k+1} 3(2^{r-1}) = \sum_{r=1}^{k} 3(2^{r-1}) + 3(2^{k+1-1})$

$= 3\,(2^k - 1) + 3\,(2^k)$
$= 3[2^k - 1 + 2^k]$
$= 3[2 \times 2^k - 1]$
$= 3\,(2^{k+1} - 1)$

by PMI $\sum_{r=1}^{n} 3(2^{r-1}) = 3(2^n - 1)$

12 RTP $n\,(n^2 + 5) = 6A,\ n \in \mathbb{Z}^+$

Proof:
when n = 1, $n\,(n^2 + 5)$
$= 1(1^2 + 5) = 6$
$= 6(1)$
Hence when n = 1, $n\,(n^2 + 5)$ is divisible by 6
Assume true for n = k, i.e. $k\,(k^2 + 5) = 6A$
RTP true for n = k +1, i.e. $(k+1)\,((k + 1)^2 + 5) = 6B$
Proof:
$(k + 1)\,((k + 1)^2 + 5)$
$= k\,((k + 1)^2 + 5) + (k + 1)^2 + 5$
$= k\,(k^2 + 2k + 6) + (k^2 + 2k + 6)$
$= k\,(k^2 + 5) + k\,(2k + 1) + k^2 + 2k + 6$
$= k\,(k^2 + 5) + 3k^2 + 3k + 6$
$= k\,(k^2 + 5) + 3k\,(k + 1) + 6$ Since k (k + 1) is the product of two consecutive integers, k (k + 1) is an even number and hence divisible by 2
$\therefore$ 3k (k + 1) is divisible by 6
$= 6A + 6C + 6$
$= 6\,(A + C + 1)$
$= 6B$
Hence by PMI $n\,(n^2 + 5)$ is divisible by 6 for all positive integers n

13 RTP $n^5 - n = 5A$

Proof:
When n = 1, $1^5 - 1 = 0$ which is divisible by 5
Hence when n = 1, $n^5 - n = 5A$
Assume true for n = k i.e. $k^5 - k = 5A$
RTP true for n = k + 1, i.e. $(k + 1)^5 - (k + 1) = 5B$
Proof:
$(k + 1)^5 - (k + 1)$
$= k^5 + 5k^4 + 10\,k^3 + 10\,k^2 + 5k \cancel{+1} - k \cancel{-1}$
$= (k^5 - k) + 5\,k^4 + 10\,k^3 + 10\,k^2 + 5\,k$
$= 5\,A + 5\,(k^4 + 2\,k^3 + 2\,k^2 + k)$
$= 5\,[A + k^4 + 2\,k^3 + 2\,k^2 + k]$
$= 5\,B$
Hence by PMI $n^5 - n$ is divisible by 5 for any positive integers n

14 RTP $\sum_{n=1}^{k}\frac{1}{(n+4)(n+3)}=\frac{1}{4}-\frac{1}{k+4}$

Proof:

k = 1, LHS = $\frac{1}{(1+4)(1+3)}=\frac{1}{5\times 4}=\frac{1}{20}$

RHS = $\frac{1}{4}-\frac{1}{5}=\frac{5-4}{20}=\frac{1}{20}$

LHS = RHS

when k = 1, $\sum_{n=1}^{k}\frac{1}{(n+4)(n+3)}=\frac{1}{4}-\frac{1}{k+4}$

Assume true for k = r, i.e. $\sum_{n=1}^{r}\frac{1}{(n+4)(n+3)}=\frac{1}{4}-\frac{1}{r+4}$

RTP true for k = r + 1, i.e. $\sum_{n=1}^{r+1}\frac{1}{(n+4)(n+3)}=\frac{1}{4}-\frac{1}{(r+1)+4}$

Proof: $\sum_{n=1}^{r+1}\frac{1}{(n+4)(n+3)}=\sum_{n=1}^{r}\frac{1}{(n+4)(n+3)}+\frac{1}{(r+5)(r+4)}$

$=\frac{1}{4}-\frac{1}{r+4}+\frac{1}{(r+5)(r+4)}$

$=\frac{1}{4}-\frac{(r+5)-1}{(r+5)(r+4)}=\frac{1}{4}-\frac{r+4}{(r+4)(r+5)}=\frac{1}{4}-\frac{1}{(r+1)+4}$