# Chapter 2 The Real Number System

**Try these 2.1**

(a) Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$

Since a and b are integers, when we multiply two integers we get an integer

$\therefore a \times b \in \mathbb{Z}$

Hence the set of integers is closed with respect to multiplication.

(b) Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$

Now $\frac{a}{b} \div \frac{c}{d}$

$= \frac{a}{b} \times \frac{d}{c}$

$= \frac{ad}{bc}$

Since $a, b, c, d \in \mathbb{Z} \Rightarrow ad \in \mathbb{Z}$ and $bc \in \mathbb{Z}$

$\therefore \frac{ad}{bc} \in \mathbb{Q}$

Hence $\mathbb{Q}$ is closed with respect to division.

(c) Let $\underline{a}, \underline{b} \in \overline{\square}$

$\underline{a} + \underline{b} \in \overline{\square}$

$\therefore \overline{\square}$ is closed with respect to addition.

**Try these 2.2**

(a) Let $a, b, c \in \mathbb{R}$.

Now $a \times (b + c)$ and $a \times b + a \times c$ give the same value.

$\therefore$ multiplication distributes over addition.

$2 + (3 \times 4) = 2 + 12 = 14.$

$(2 + 3) \times (2 + 4) = 5 \times 6 = 30.$

$14 \neq 30$, $\therefore$ addition does not distribute over multiplication.

**Try these 2.3**

(a) Let $a, e \in \mathbb{N}$

$\frac{a}{e} = a$

$\Rightarrow e = \frac{a}{a} = 1$

$\therefore$ the identity for division of Natural numbers is 1

(b) Let $a, e \in \mathbb{R}$

Now $a * e = a + 2e + 4$

If e is the identity then $a * e = a$

$\Rightarrow a + 2e + 4 = a$

$2e + 4 = 0$

$2e = -4$

$\Rightarrow e = -2$

$\therefore$ There is an identity element which is –2

**Try these 2.4**

Let $a, b \in \mathbb{Z}$

If b is the inverse of a with respect to multiplication then $ab = 1$

$\Rightarrow a = \frac{1}{b}$

$\frac{1}{b} \in \mathbb{Z}$ only if $b = 1$

$\Rightarrow a = \frac{1}{1} = 1$

Since the set is the set of integers the only element that has an inverse with respect to multiplication is the identity 1.

**Exercise 2**

**1** Since 2 is a prime number and 2 is not odd
$\Rightarrow$ the statement is not true.

**2** RTP if $x = 4n$ then $x = (a)^2 - (b)^2$
Proof:
Since x is divisible by 4
$\Rightarrow x = 4n = n^2 + 2n + 1 - n^2 + 2n - 1$
$= (n + 1)^2 - (n - 1)^2$
$= a^2 - b^2$ where $a = n + 1$, $b = n - 1$
Hence if x is an integer divisible by 4, then x is the difference of two squares

**3** 6 is an even number but $2 \times 3 = 6$
2 is even and 3 is odd
The statement is false

**4** R.T.P if $x, y \in \mathbb{R}$, $x^2 + y^2 \geq 2xy$
Proof:
Since x, y are real
$(x - y)^2 \geq 0$
$\Rightarrow x^2 - 2xy + y^2 \geq 0$
$\Rightarrow x^2 + y^2 \geq 2xy$

**5** The statement in false
Let $x = \frac{1}{4}$, $\sqrt{\frac{1}{4}} = \frac{1}{2}$
Since $\frac{1}{2} > \frac{1}{4}$
$\Rightarrow \sqrt{\frac{1}{4}} > \frac{1}{4}$

**6** Let $a, b \in \mathbb{R}$. $ab = 0 \Leftrightarrow a = 0$ or $b = 0$
Proof:
Suppose that $ab = 0$. Then either $a = 0$ or $a \neq 0$
If $a = 0 \Rightarrow a = 0$ or $b = 0$
if $a \neq 0$, then $a^{-1}$ exists

$\therefore$ ab = 0
$\Rightarrow a^{-1}(ab) = a^{-1}0$
$\Rightarrow (a^{-1}a)b = 0$
$\Rightarrow 1b = 0$
b = 0
$\Rightarrow$ b = 0, since any number multiplied by 0 gives 0
Hence if ab = 0 $\Rightarrow$ a = 0 or b = 0
Now if a = 0 $\Rightarrow$ ab = 0b
$\Rightarrow$ab = 0
If b = 0, $\Rightarrow$ ab = a0
$\Rightarrow$ ab = 0
Hence if a = 0 or b = 0 then ab = 0
$\therefore$ ab = 0 $\Leftrightarrow$ a = 0 or b = 0

**7** $\frac{1}{(a+b)^2}$

Let a = 2, b = 4

$\frac{1}{(2+4)^2} = \frac{1}{36}$

$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{2^2} + \frac{1}{4^2}$

$= \frac{1}{4} + \frac{1}{16}$

$= \frac{4+1}{16}$

$= \frac{5}{16}$

Since $\frac{1}{36} \neq \frac{5}{16}$

$\Rightarrow \frac{1}{(a+b)^2} \neq \frac{1}{a^2} + \frac{1}{b^2}$ when a = 2, b = 4

Hence $\frac{1}{(a+b)^2}$ is not equivalent to $\frac{1}{a^2} = \frac{1}{b^2}$

**8** $a * b = a + b + 5$

(a) Let a, b $\in \mathbb{R}$
Since a and b are real numbers
a + b is also real
a+b+5 is real
$\Rightarrow a + b + 5 \in \mathbb{R}$
$\therefore\ a * b \in$ □ .
Hence $\mathbb{R}$ is closed wrt *

(b) Let e be the identity:
Now $a * e = a$
$\Rightarrow$ a + e + 5 = a
e + 5 = 0
e = – 5
$\therefore$ the identity is –5
For the inverse of a
$a * a^{-1} = e$

 

$\therefore\ a + a^{-1} + 5 = -5$

$a^{-1} = -10 - a$

the inverse of a is – 10 – a

**9** $a * b = 3(a + b)$

a ■ b = 2ab

Let a, b, c ∈ R

$(a * b) * c = 3(a + b) * c$

= 3[3a + 3b + c]

= 9a + 9b + 3c

$a * (b * c) = a * 3(b + c)$

$= a * (3b + 3c)$

= 3[a + 3b + 3c]

= 3a + 9b + 9c

Since $(a * b) * c \neq a * (b * c)$

⇒ * is not associative

Now (a ■ b) ■ c = 2ab ■ c

= 2(2ab)(c)

= 4 abc

and a ■ (b ■ c) = a ■ (2bc)

= 2 a (2 bc)

= 4 abc

∴ (a ■ b) ■ c = a ■ (b ■ c)

⇒ ■ is associative

**10** $2^n + 3$

When n = 5, $2^n + 3$

$= 2^5 + 3$

= 32 + 3

= 35

Since 35 is not prime, the statement is false

**11** (a) Since a and b are real

a – b is real

and | a – b | is real and positive

∴ For every a, b ∈ A, | a – b| ∈ A

⇒ A is closed with respect to ∗

(b) Let e be the identity

a ∗ e = a

| a – e | = a

⇒ e = 0 since a ≥ 0

The identity is 0

(c) If a has an inverse then $a * a^{-1} = e$

$\Rightarrow |a - a^{-1}| = |0| = 0,\ a - a^{-1} = 0,\ a^{-1} = a$

Hence the inverse of a is a

∴ the elements are self inverses

(d) $(1 * 2) * 3 = ||1 - 2| - 3| = |1 - 3| = 2$

$1 * (2 * 3) = |1 - |2 - 3|| = |1 - 1| = 0$

so by counterexample ∗ is not associative

**12** $a\ \Delta\ b = a^{\ln b}$

(a) R.T.S. $(a\ \Delta\ b)\ \Delta c = a\ \Delta(b\ \Delta\ c)\ \forall\ a, b, c \in \mathbb{R}^+$

Solution:

$(a\ \Delta\ b)\ \Delta\ c = (a^{\ln b})\ \Delta\ c$

$= (a^{\ln b})^{\ln c} = a^{\ln b \times \ln c}$

Also $a \Delta (b \Delta c) = a \Delta (b^{\ln c}) = a^{\ln b^{\ln c}} = a^{\ln c \ln b}$

$\therefore (a \Delta b) \Delta c = a \Delta (b \Delta c)$

(b) Let a, b, $\in \mathbb{R}^+$

Now $a \Delta b = a$

$\therefore a^{\ln b} = a$

$\ln a^{\ln b} = \ln a$

$\ln b \ln a = \ln a$

$\ln b = 1$

$b = e$

$\therefore$ the identity is e

(c) Let a, b, c $\in \mathbb{R}^+$

$a \Delta (b \times c) = a \Delta bc = a^{\ln bc}$

and $(a \Delta b) \times (a \Delta c) = a^{\ln b} \times a^{\ln c} = a^{\ln b + \ln c} = a^{\ln bc}$

Since $a \Delta (b \times c) = (a \Delta b) \times (a \Delta c)$

$\Rightarrow \Delta$ distributes over multiplication

**13** (a) Let x, y, $x_1$, $y_1$, $\in \mathbb{R}$

$(x + y\sqrt{3}) + (x_1 + y_1\sqrt{3})$

$= (x + x_1) + y\sqrt{3} + y_1\sqrt{3}$

$= (x + x_1) + (y + y_1)\sqrt{3}$

Since x, $x_1$, y, $y_1$, are integers

$\Rightarrow x + x_1$ and $y + y_1$ are integers

$\therefore (x + x_1) + (y + y_1)\sqrt{3} \in X$

Hence X is closed under addition

$(x + y\sqrt{3})(x_1 + y_1\sqrt{3})$

$= xx_1 + xy_1\sqrt{3} + x_1 y\sqrt{3} + 3yy_1$

$= (xx_1 + 3yy_1) + [xy_1 + x_1 y]\sqrt{3}$

Since x, $x_1$, y, $y_1$, are integers

$\Rightarrow xx_1, yy_1, xy_1$ and $x_1y$ are integers

$\therefore (xx_1 + 3yy_1) + [xy_1 + x_1 y]\sqrt{3} \in X$

Hence X is closed under multiplication

(b) Let $e_1, e_2 \in \mathbb{R}$

$(x + y\sqrt{3}) + (e_1 + e_2\sqrt{3}) = x + y\sqrt{3}$

$\Rightarrow e_1 + e_2\sqrt{3} = 0 + 0\sqrt{3}$

$\Rightarrow e_1 = 0, e_2 = 0$

$\therefore$ the identity with respect to addition is $0 + 0\sqrt{3}$

(c) $(x + y\sqrt{3})(e_1 + e_2\sqrt{3}) = x + y\sqrt{3}$

$e_1 + e_2\sqrt{3} = 1$

$e_1 = 1, e_2 = 0$

$\therefore$ identity with respect to multiplication is 1

(d) For $a \in X$, inverse of $a = \dfrac{1}{a}$

For $a = 0 + 0\sqrt{3} = 0 \in X$,

inverse of $a = \dfrac{1}{0} \notin X$

$\therefore$ not every element of X has an inverse with respect to multiplication

**14)** $x \Delta y = \sqrt{x^2 + y^2}$

(a) $x, y, z \in \mathbb{R}$

$(x \Delta y) \Delta z = \sqrt{x^2 + y^2} \Delta z$

$= \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2}$

$= \sqrt{x^2 + y^2 + z^2}$

$x \Delta (y \Delta z) = x \Delta \sqrt{y^2 + z^2}$

$= \sqrt{x^2 + \left(\sqrt{y^2 + z^2}\right)^2}$

$= \sqrt{x^2 + y^2 + z^2}$

$\therefore$ $(x \Delta y) \Delta z = x \Delta (y \Delta z)$

$\Rightarrow \Delta$ is associative

(b) Let e be the identity:

$x \Delta e = x$

$\sqrt{x^2 + e^2} = x$

$e^2 = 0$, $e = 0$

$\therefore$ the identity is 0

(c) $x \Delta (y \Delta z) = x \Delta \sqrt{y^2 + z^2}$

$= \sqrt{x^2 + y^2 + z^2}$

$(x \Delta y) \Delta (x \Delta z) = \sqrt{x^2 + y^2} \Delta \sqrt{x^2 + z^2}$

$= \sqrt{x^2 + y^2 + x^2 + z^2}$

$= \sqrt{2x^2 + y^2 + z^2}$

$\Rightarrow x \Delta (y \Delta z) \neq (x \Delta y) \Delta (x \Delta z)$

$\Delta$ does not distribute over $\Delta$

**15** $a * b = a + b - ab,\ a, b \in R$

(a) $a * e = a$

$a * e = a + e - ea$

$a + e - ea = a$

$e(1 - a) = 0$

$e = 0$

identity is 0

(b) $a * a^{-1} = e$

$a + a^{-1} - aa^{-1} = 0$

$a = aa^{-1} - a^{-1}$

$= (a - 1)a^{-1}$

$a^{-1} = \dfrac{a}{a - 1}$

$\therefore$ the inverse of a is $\dfrac{a}{a-1}$, $a \neq 1$

(c) $a * b = a + b - ab$

$b * a = b + a - ba = a + b - ab$

Since $a * b = b * a$

$*$ is commutative

(d) $a * (a * 2) = 10$

$\Rightarrow a * (a + 2 - 2a) = 10$

$a*(2-a)=10$

$a + 2 - a - a(2 - a) = 10$

$2-2a+a^2=10$

$a^2 - 2a - 8 = 0$

$(a - 4)(a + 2) = 0$

$a = 4, -2$

**16** (a) Since all elements in the table belong to S

$\Rightarrow$ S in closed with respect to $\Delta$

(b) identity is a

(c)

| Element | inverse |
|---|---|
| a | a |
| b | c |
| c | b |
| d | d |

**17** (a) Identity is q

(b)

| Element | Inverse |
|---|---|
| p | r |
| q | q |
| r | p |
| s | s |