Chapter 17 Differential Equations

Exercise 17

1 $\frac{dy}{dx} = 4x^3 - 2x + 1$

$y = \int 4x^3 - 2x + 1\,dx$

$y = x^4 - x^2 + x + c$

2 $\frac{dy}{dx} = \sin\left(\frac{1}{2}x\right)$

$y = \int \sin\left(\frac{1}{2}x\right)dx$

$y = -2\cos\left(\frac{1}{2}x\right) + c$

3 $\frac{dy}{dx} = \cos^2 x$

$y = \int \cos^2 x\,dx$

$y = \frac{1}{2}\int 1 + \cos 2x\,dx$

$y = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + c$

4 $\frac{dy}{dx} = x(x^2 + 2)$

$y = \int x^3 + 2x\,dx$

$y = \frac{1}{4}x^4 + x^2 + c$

5 $\frac{dy}{dx} = \frac{y^2}{\sqrt{3x+1}}$

$\int \frac{1}{y^2}dy = \int (3x+1)^{-1/2}dx$

$\Rightarrow -y^{-1} = \frac{1}{3}\frac{(3x+1)^{1/2}}{1/2} + c$

$\Rightarrow -\frac{1}{y} = \frac{2}{3}(3x+1)^{1/2} + c$

$y = \frac{1}{-\frac{2}{3}(3x+1)^{1/2} - c}$

6 $x^2\frac{dy}{dx} = y^2$

$\int \frac{1}{y^2}dy = \int \frac{1}{x^2}dx$

$\int y^{-2}dy = \int x^{-2}dx$

$$\frac{y^{-1}}{-1}=\frac{x^{-1}}{-1}+c$$

$$-\frac{1}{y}=-\frac{1}{x}+c$$

$$=\frac{-1+cx}{x}$$

$$y=-\left(\frac{x}{-1+cx}\right)$$

7 $\sec^2 x\,\frac{dy}{dx}=\cos^2 x$

$$\frac{dy}{dx}=\cos^4 x$$

$$y=\int \cos^4 x\,dx$$

$$y=\int\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$y=\frac{1}{4}\int 1+2\cos 2x+\cos^2 2x\,dx$$

$$y=\frac{1}{4}\int 1+2\cos 2x+\frac{1+\cos 4x}{2}\,dx$$

$$y=\frac{1}{4}\int \frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\,dx$$

$$y=\frac{1}{4}\left[\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right]+c$$

8 $\frac{dy}{dx}=y^2+2xy^2$

$$\int\frac{1}{y^2}\,dy=\int(1+2x)\,dx$$

$$-\frac{1}{y}=x+x^2+c$$

$$y=\frac{1}{-x^2-x-c}$$

9 $\frac{dx}{dt}=\cos t+\sin t$

$$x=\int \cos t+\sin t\,dt$$

$x=\sin t-\cos t+c$

$x=0,\ t=0\Rightarrow 0=-1+c,\ c=1$

$x=\sin t-\cos t+1$

10 $\frac{dy}{dx}=(x+1)^3$

$$y=\int(x+1)^3\,dx$$

$$y=\frac{1}{4}(x+1)^4+c$$

$$y=0,\ x=2\Rightarrow 0=\frac{1}{4}(3)^4+c$$

$c = -\frac{81}{4}$

$y = \frac{1}{4}(x+1)^4 - \frac{81}{4}$

11 $\frac{dy}{dt} = \sin 2t$

$y = \int \sin 2t \, dt$

$y = \frac{-1}{2}\cos 2t + c$

$y = 1, t = \pi/4 \Rightarrow 1 = -\frac{1}{2}\cos\frac{\pi}{2} + c$

$c = 1$

$y = -\frac{1}{2}\cos 2t + 1$

12 $\frac{dy}{dx} = 2xy^2 - y^2$

$\int \frac{1}{y^2} dy = \int 2x - 1 \, dx$

$-\frac{1}{y} = x^2 - x + c$

$y = \frac{1}{x^2 - x + c}$

13 $\frac{dy}{dx} = \frac{(4x+1)^3}{(y-1)^2}$

$\int (y-1)^2 \, dy = \int (4x+1)^3 \, dx$

$\Rightarrow \frac{(y-1)^3}{3} = \frac{1}{16}(4x+1)^4 + c$

$y - 1 = \sqrt[3]{\frac{3}{16}(4x+1)^4 + c}$

$y = 1 + \sqrt[3]{\frac{3}{16}(4x+1)^4 + c}$

14 $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}$

$\int (2y-4) \, dy = \int 3x^2 + 4x - 4 \, dx$

$\Rightarrow \frac{1}{4}(2y-4)^2 = x^3 + 2x^2 - 4x + c$

$x = 3, y = 1 \Rightarrow 1 = 27 + 18 - 12 + c$

$c = -32$

$\frac{1}{4}(2y-4)^2 = x^3 + 2x^2 - 4x - 32$

$(2y-4)^2 = 4x^3 + 8x^2 - 16x - 128$

$2y - 4 = \sqrt{4x^3 + 8x^2 - 16x - 128}$

$y = \frac{1}{2}\sqrt{4x^3 + 8x^2 - 16x - 128} + 2$

15 $I\frac{dw}{dt} = -\alpha$

$\int I\,dw = \int -\alpha\,dt$

$Iw = -\alpha t + c$

$t = 0,\ w = w_0 \Rightarrow I\,w_0 = c$

$I\,w = -\alpha t + I\,w_0$

$w = \frac{-\alpha}{I}t + w_0$

16 $\frac{dv}{dt} = -kv^2$

$\int \frac{1}{v^2}\,dv = \int -k\,dt$

$-\frac{1}{v} = -kt + c$

$t = 0,\ v = v_1 \Rightarrow -\frac{1}{v_1} = c$

$-\frac{1}{v} = -kt - \frac{1}{v_1}$

$\frac{1}{v} = \frac{ktv_1 + 1}{v_1}$

$v = \frac{v_1}{ktv_1 + 1}$

17 $\frac{dx}{dt} \propto (9 - x)^{1/3}$

$\frac{dx}{dt} = k(9 - x)^{1/3}$

$x = 1,\ \frac{dx}{dt} = 1$

$\Rightarrow 1 = k\,(8)^{1/3}$

$k = \frac{1}{2}$

$\frac{dx}{dt} = \frac{1}{2}(9 - x)^{1/3}$

$\int \frac{2}{(9 - x)^{1/3}}\,dx = \int dt$

$\Rightarrow \int 2\,(9 - x)^{-1/3}\,dx = \int dt$

$\Rightarrow -2\,\frac{(9 - x)^{2/3}}{2/3} = t + c$

$\Rightarrow -3(9 - x)^{2/3} = t + c$

$t = 0,\ x = 1 \Rightarrow -3(8)^{2/3} = c$

$-12 = c$

$-3(9 - x)^{2/3} = t - 12$

$(9 - x)^{2/3} = \frac{t - 12}{-3}$

$$9 - x = \left(\frac{12-t}{3}\right)^{3/2}$$

$$x = 9 - \left(\frac{12-t}{3}\right)^{3/2}$$

18 (a) $-\frac{dp}{dt} \propto \sqrt{p - p_0}$

$$-\frac{dp}{dt} = k\sqrt{p - p_0}$$

(b) $\int -\frac{1}{\sqrt{p - p_0}}\,dp = \int k\,dt$

$$\int -(p - p_0)^{-1/2}\,dp = \int k\,dt$$

$$-2(p - p_0)^{1/2} = kt + c$$

(c) $t = 0,\ p = 5p_0$

$$-2\,(4p_0)^{1/2} = c$$

$$c = -4\sqrt{p_0}$$

$$-2\,(p - p_0)^{1/2} = kt - 4\sqrt{p_0}$$

$$t = 2,\ p = 2p_0 \Rightarrow -2\sqrt{p_0} = 2k - 4\sqrt{p_0}$$

$$2\sqrt{p_0} = 2k$$

$$k = \sqrt{p_0}$$

(d) $-2(p - p_0)^{1/2} = t\sqrt{p_0} - 4\sqrt{p_0}$

$$(p - p_0)^{1/2} = -\frac{1}{2}t\sqrt{p_0} + 2\sqrt{p_0}$$

$$p - p_0 = \left[\sqrt{p_0}\left(2 - \frac{1}{2}t\right)\right]^2$$

$$p = p_0 + p_0\left(2 - \frac{1}{2}t\right)^2$$

19 (a) $\frac{dy}{dt} \propto (40 - y)^2$

$$\frac{dy}{dt} = k(40 - y)^2$$

(b) $y = 0,\ \frac{dy}{dt} = 4 \Rightarrow 4 = k(40)^2$

$$k = \frac{4}{1600} = \frac{1}{400}$$

$$\frac{dy}{dt} = \frac{1}{400}(40 - y)^2$$

$$\frac{1}{(40 - y)^2}\,dy = \frac{1}{400}\,dt$$

$$\int \frac{1}{(40 - y)^2}\,dy = \int \frac{1}{400}\,dt$$

$$\Rightarrow \frac{1}{40 - y} = \frac{1}{400}t + c$$

$$40 - y = \frac{1}{\frac{t}{400}+c}$$

$$y = 40 - \frac{1}{\frac{t}{400}+c}$$

(c) $t = 0, y = 0 \Rightarrow 0 = 40 - \frac{1}{c}$

$$\frac{1}{c} = 40$$

$$c = \frac{1}{40}$$

$$y = 40 - \frac{1}{\frac{t}{400}+\frac{1}{40}} = 40 - \frac{400}{t+10}$$

$$y = 35 \Rightarrow 35 = 40 - \frac{400}{t+10}$$

$$5 = \frac{400}{t+10}$$

$$t + 10 = \frac{400}{5} = 80 \text{ cm}$$

t = 70 minutes

20 (a) $\frac{dx}{dt} \propto \frac{1}{x^2}$

$$\frac{dx}{dt} = \frac{k}{x^2}$$

$$\int x^2 dx = \int k\, dt$$

$$\frac{1}{3}x^3 = kt + c$$

(b) $t = 0, x = 0 \Rightarrow c = 0$

$$\frac{1}{3}x^3 = kt$$

$$t = 2, x = 18 \Rightarrow \frac{18^3}{3} = 2k$$

$$k = \frac{18^3}{3} \times \frac{1}{2} = 972$$

$$\therefore \frac{1}{3}x^3 = 972t$$

$$x^3 = 2916t$$

$$x = 30 \Rightarrow \frac{30^3}{3} = 2916t$$

t = 3.09

21 (a) $\frac{d^2y}{dx^2} = (3x+2)^2$

$$\Rightarrow \int \frac{d^2y}{dx^2} dx = \int (3x + 2)^2\, dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{9}(3x+2)^3 + A$$

$$\int \frac{dy}{dx}\,dx = \int \frac{1}{9}(3x+2)^3 + A\,dx$$

$$y = \frac{1}{108}(3x+2)^4 + Ax + B$$

(b) $\frac{d^2y}{dx^2} = \sin 2x \cos 2x$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\sin 4x \text{, since } \sin 4x = 2\sin 2x \cos 2x$$

$$\int \frac{d^2y}{dx^2}\,dx = \int \left(\frac{1}{2}\sin 4x\right)dx$$

$$\frac{dy}{dx} = -\frac{1}{8}\cos 4x + A$$

$$\int \frac{dy}{dx}\,dx = \int \left(-\frac{1}{8}\cos 4x + A\right)dx$$

$$y = -\frac{1}{32}\sin 4x + Ax + B$$

22 $\frac{d^2y}{dt^2} = -4t^2 + 5t + 3$

$$\Rightarrow \int \frac{d^2y}{dt^2}\,dt = \int (-4t^2 + 5t + 3)\,dt$$

$$\Rightarrow \frac{dy}{dt} = -\frac{4}{3}t^3 + \frac{5}{2}t^2 + 3t + A$$

$$\int \frac{dy}{dt}\,dt = \int \left(-\frac{4}{3}t^3 + \frac{5}{2}t^2 + 3t + A\right)dt$$

$$\Rightarrow y = -\frac{1}{3}t^4 + \frac{5}{6}t^3 + \frac{3}{2}t^2 + At + B$$

When $t = 0$, $y = 1 \Rightarrow 1 = B$

$$y = -\frac{1}{3}t^4 + \frac{5}{6}t^3 + \frac{3}{2}t^2 + At + 1$$

$$\frac{dy}{dt} = \frac{-4}{3}t^3 + \frac{5}{2}t^2 + 3t + A$$

$$t = 0,\ \frac{dy}{dt} = 1 \Rightarrow 1 = A$$

$$\therefore y = -\frac{1}{3}t^4 + \frac{5}{6}t^3 + \frac{3}{2}t^2 + t + 1$$

23 $\frac{d^2y}{dx^2} = x + 2$

$$\int \frac{d^2y}{dx^2}\,dx = \int (x+2)\,dx$$

$$\frac{dy}{dx} = \frac{1}{2}(x+2)^2 + A$$

$$\int \frac{dy}{dx} = \int \frac{1}{2}(x+2)^2 + A\,dx$$

$\Rightarrow y = \frac{1}{6}(x+2)^3 + Ax + B$

$x = 0,\ y = 0 \Rightarrow 0 = \frac{8}{6} + B$

$B = \frac{-8}{6} = \frac{-4}{3}$

$\therefore y = \frac{1}{6}(x+2)^3 + Ax - \frac{4}{3}$

$\frac{dy}{dx} = \frac{1}{2}(x+2)^2 + A$

$x = 0,\ \frac{dy}{dx} = 1 \Rightarrow 1 = 2 + A$

$A = -1$

$\therefore y = \frac{1}{6}(x+2)^3 - x - \frac{4}{3}$

$\frac{dy}{dx} = \frac{1}{2}(x+2)^2 - 1 = 0$

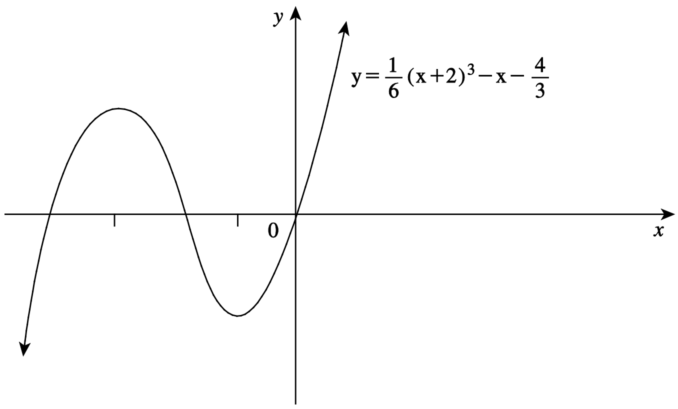
$\Rightarrow \frac{1}{2}(x+2)^2 = 1$

$x + 2 = \pm\sqrt{2}$

$x = -2 \pm \sqrt{2}$

when $x = -2 + \sqrt{2}$, $\frac{d^2y}{dx^2} = -2 + \sqrt{2} + 2 > 0$ min point

$x = -2 - \sqrt{2}, \frac{d^2y}{dx^2} = -2 - \sqrt{2} + 2 < 0$ max point



24 $\frac{dx}{dt} = \frac{2}{(1-2x)^3}$

$\int (1-2x)^3\,dx = \int 2\,dt$

$-\frac{1}{8}(1-2x)^4 = 2t + c$

$(1-2x)^4 = -16t + c$

$1 - 2x = \sqrt[4]{-16t + c}$

$x = \frac{1}{2}\left[1 - \sqrt[4]{-16t + c}\right]$

25 (a) $\frac{d^2x}{dt^2} = 13 - 6t$

$\int \frac{d^2x}{dt^2}\,dt = \int (13 - 6t)\,dt$

$\Rightarrow \frac{dx}{dt} = 13t - 3t^2 + A$

when $t = 0$, $\frac{dx}{dt} = 30$

$\Rightarrow 30 = A$

$\therefore \frac{dx}{dt} = -3t^2 + 13t + 30$

When $t = 3$, $\frac{dx}{dt} = -3(3)^2 + 13(3) + 30 = 42$

(b) when $\frac{dx}{dt} = 0 \Rightarrow -3t^2 + 13t + 30 = 0$

$3t^2 - 13t - 30 = 0$

$(3t + 5)(t - 6) = 0$

$t = \frac{-5}{3}, t = 6$

Since $t > 0$, $t = 6$s

(c) Now $\frac{dx}{dt} = -3t^2 + 13t + 30$

$\Rightarrow \int \frac{dx}{dt}\,dt = \int (-3t^2 + 13t + 30)\,dt$

$\Rightarrow x = -t^3 + \frac{13}{2}t^2 + 30t + B$

$t = 0, x = 0 \Rightarrow 0 = B$

$\therefore x = -t^3 + \frac{13}{2}t^2 + 30t$

$t = 6 \Rightarrow x = -216 + 234 + 180 = 198$m