Chapter 16 Applications of Integration

Exercise 16A

1 (a) $x^2 - 4x + 7 = 3x - 5$

$x^2 - 7x + 12 = 0$

$(x - 3)(x - 4) = 0$

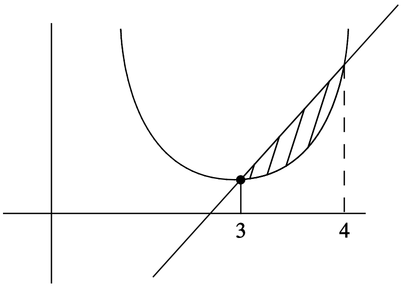
$x = 3, 4$

$x = 3, y = 9 - 5 = 4$

$x = 4, y = 12 - 5 = 7$

P (3, 4) Q (4, 7)

(b)



Area under curve $= \int_3^4 x^2 - 4x + 7 \, dx$

$= \left[\frac{1}{3}x^3 - 2x^2 + 7x\right]_3^4$

$= \left(\frac{64}{3} - 32 + 28\right) - (9 - 18 + 21)$

$= \frac{64}{3} - \frac{12}{3} - \frac{36}{3} = \frac{16}{3}$

Required area $= \frac{1}{2}(1)(7 + 4) - \frac{16}{3}$

$= \frac{11}{2} - \frac{16}{3}$

$= \frac{33 - 32}{6} = \frac{1}{6}$

2 $y = \sin x$, $y = \cos x$

$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4}$

Calculate as 2 regions – A: area under $y = \sin x$ for $x = 0$ to $x = \frac{\pi}{4}$

B: area under $y = \cos x$ for $x = \frac{\pi}{4}$ to $\frac{\pi}{2}$

Shaded area $= \int_0^{\pi/4} \sin x \, dx + \int_{\pi/4}^{\pi/2} \cos x \, dx$

$= [-\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$

$= -\frac{\sqrt{2}}{2} + 1 + 1 - \frac{\sqrt{2}}{2}$

$= 2 - \sqrt{2}$

3 $\int_{-1}^{2} 4 - x^2 \, dx$

$= \left[4x - \frac{1}{3}x^3 \right]_{-1}^{2}$

$= \left(8 - \frac{8}{3}\right) - \left(-4 + \frac{1}{3}\right)$

$= 12 - 3$

$= 9$

4 $\int_{-2}^{3} (x + 2)(3 - x) \, dx$

$= \int_{-2}^{3} x - x^2 + 6 \, dx$

$= \left[6x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^{3}$

$= \left(18 + \frac{9}{2} - 9\right) - \left(-12 + 2 + \frac{8}{3}\right)$

$= 19 + \frac{9}{2} - \frac{8}{3} = \frac{125}{6}$

5 $y = 7x$

$y = 9x - x^2$

$9x - x^2 = 7x$

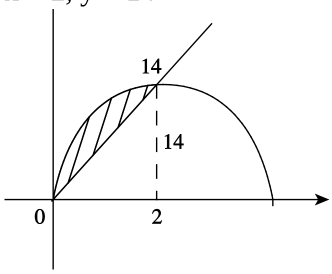
$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0, 2$

when $x = 0, y = 0$ P (2, 14)

$x = 2, y = 14$



Total area $= \int_{0}^{9} 9x - x^2 \, dx$

$= \left[\frac{9}{2}x^2 - \frac{1}{3}x^3 \right]_{0}^{9}$

$= \frac{729}{2} - \frac{729}{3}$

$= \frac{243}{2}$

Area of A $= \int_{0}^{2} 9x - x^2 \, dx - \frac{2 \times 14}{2}$

$= \left[\frac{9}{2}x^2 - \frac{1}{3}x^3\right]_0^2 - 14$

$= 18 - \frac{8}{3} - 14$

$= \frac{4}{3}$

Area of B $= \frac{243}{2} - \frac{4}{3}$

$= \frac{721}{6}$

Ratio $= \frac{\frac{4}{3}}{\frac{721}{6}} = \frac{8}{721}$

Ratio 8 : 271

6 $4x - x^2 = 0$

$x(4 - x) = 0$

$x = 0, 4$

$\int_0^4 4x - x^2\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4$

$= 32 - \frac{64}{3}$

$= \frac{96 - 64}{3} = \frac{32}{2}$ units^2

7 $y = x + \frac{4}{x^2}$

Required area $= \int_1^3 x + \frac{4}{x^2}\,dx$

$= \left[\frac{1}{2}x^2 - \frac{4}{x}\right]_1^3$

$= \left(\frac{9}{2} - \frac{4}{3}\right) - \left(\frac{1}{2} - 4\right)$

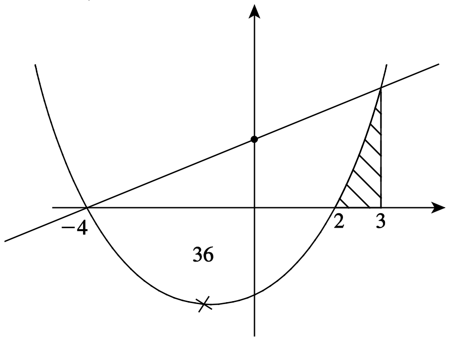
$= \frac{20}{3}$

8 $x^2 + 2x - 8 = x + 4$

$x^2 + x - 12 = 0$

$(x + 4)(x - 3) = 0$

$x = -4, 3$



Area under the curve from – 4 to 2 $= \left|\int_{-4}^{2} x^2 + 2x - 8\,dx\right|$

$= \left|\left[\frac{1}{3}x^3 + x^2 - 8x\right]_{-4}^{2}\right|$

$= \left|\left(\frac{8}{3} + 4 - 16\right) - \left(\frac{-64}{3} + 16 + 32\right)\right|$

$= |-36| = 36$

Area under the curve from 2 to 3 $= \int_{2}^{3} x^2 + 2x - 8\,dx$

$= \left[\frac{1}{3}x^3 + x^2 - 8x\right]_{2}^{3}$

$= (9 + 9 - 24) - \left(\frac{8}{3} + 4 - 16\right)$

$= -6 + 12 - \frac{8}{3} = \frac{10}{3}$

Area of $\Delta = \frac{7 \times 7}{2} = \frac{49}{2}$

Required area $= 36 + \frac{49}{2} - \frac{10}{3} = \frac{343}{6}$

9 (a) y = 2 + sin 3x

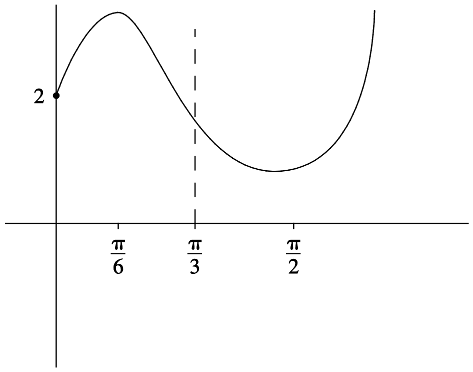
$\frac{dy}{dx} = 3\cos 3x$

3 cos 3x = 0

$\Rightarrow$ cos 3x = 0

$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}$



(b) Area $= \int_{0}^{\frac{\pi}{3}} 2 + \sin 3x\,dx$

$= \left[2x - \frac{1}{3}\cos 3x\right]_{0}^{\pi/3}$

$= \left(\frac{2\pi}{3} - \frac{1}{3}\cos\pi\right) - \left(-\frac{1}{3}\right)$

$= \frac{2\pi}{3} + \frac{2}{3}$

10 Area $= \int_0^{\pi/4} 2\cos 2x \, dx$

$= [\sin 2x]_0^{\pi/4}$

$= 1$

11 (a) $y = \frac{1}{x^2} + 16x$

$\frac{dy}{dx} = \frac{-2}{x^3} + 16$

$\frac{dy}{dx} = 0 \Rightarrow \frac{2}{x^3} = 16$

$x^3 = \frac{1}{8}$

$x = \frac{1}{2}$

(b) Area under the curve from $\frac{1}{4}$ to $\frac{1}{2} = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{x^2} + 16x \, dx$

$= \left[-\frac{1}{x} + 8x^2 \right]_{1/4}^{1/2}$

$= (-2 + 2) - \left(-4 + \frac{1}{2} \right)$

$= 3\frac{1}{2}$

A = $\left(\frac{1}{2}, 12 \right)$, gradient of OA = $\frac{12}{\frac{1}{2}}$ = 24

equation of line OA: y = 24 x

When x = $\frac{1}{4}$, y = 6

Area of trapezia $= \frac{1}{2}\left(\frac{1}{4}\right)(12 + 6)$

$= \frac{9}{4}$.

Shaded Area $= 3\frac{1}{2} - \frac{9}{4} = \frac{5}{4}$

12 Required area $= \int_0^{\pi/8} 6\cos^2(2x) \, dx$

$= 6\int_0^{\pi/8} \frac{1 + \cos 4x}{2} \, dx$

$= 3\int_0^{\pi/8} (1 + \cos 4x) \, dx$

$= 3\left[x + \frac{1}{4}\sin 4x \right]_0^{\pi/8}$

$= 3\left[\frac{\pi}{8} + \frac{1}{4}\sin\left(\frac{\pi}{2}\right) \right]$

$= 3\left[\frac{\pi}{8} + \frac{1}{4}\right]$

13 $x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$

$= (x - 1)(x - 2)(x - 3)$

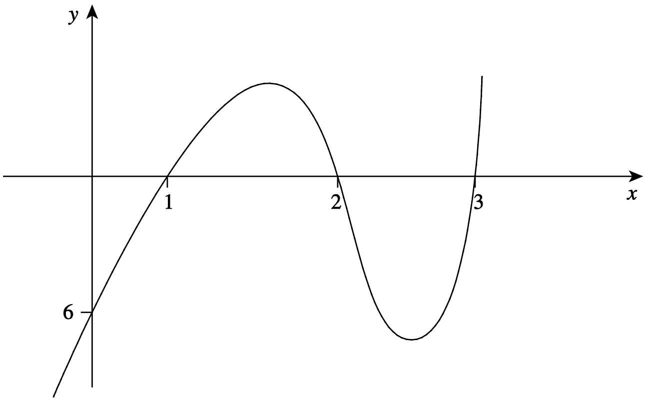
$\frac{dy}{dx} = 3x^2 - 12x + 11$

$3x^2 - 12x + 11 = 0$

$x = \frac{12 \pm \sqrt{144 - 132}}{6}$

$= 1.43, 2.58$

$x = 0, y = -6$



$\text{Area } = \int_1^2 x^3 - 6x^2 + 11x - 6\, dx + \left|\int_2^3 x^3 - 6x^2 + 11x - 6)\, dx\right|$

$= \left[\frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x\right]_1^2 + \left|\left[\frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x\right]_2^3\right|$

$= (4 - 16 + 22 - 12) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right)$

$+ \left|\left[\frac{81}{4} - 54 + \frac{99}{2} - 18\right] - [4 - 16 + 22 - 12]\right|$

$= -2 - (-2.25) + |-2.25 - (-2)|$

$= \frac{1}{2}$

14 $y = 2x^3 + 3x^2 - 23x - 12$

$2x^3 + 3x^2 - 23x - 12$

$= (2x + 1)(x^2 + x - 12)$

$= (2x + 1)(x + 4)(x - 3)$

$\frac{dy}{dx} = 6x^2 + 6x - 23 = 0 \quad -2 + 3 + 23 - 12$

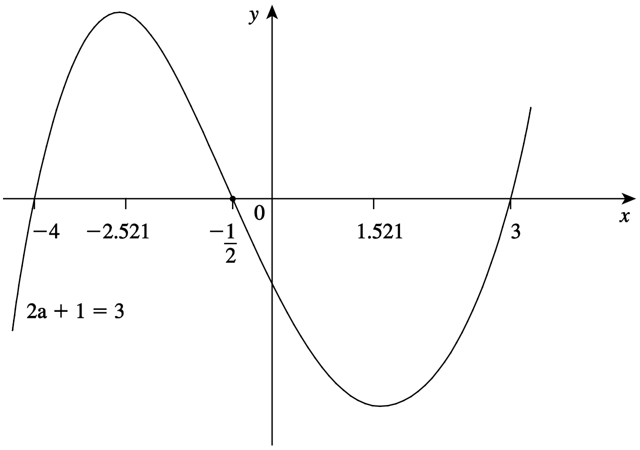
$x = \frac{-6 \pm \sqrt{36 - 4(6)(-23)}}{2(6)}$

$= \frac{-6 \pm 14\sqrt{3}}{12}$

$= 1.521 \;, \; -2.521$

$x = 1.521, y = 2(1.521)^3 + 3(1.521)^2 - 23(1.521) - 12$

$= -33.01$



$$\text{Area} = \int_{-4}^{-1/2} 2x^3 + 3x^2 - 23x - 12\,dx + \left|\int_{-\frac{1}{2}}^{3} 2x^3 + 3x^2 - 23x - 12\,dx\right|$$

$$= \left[\frac{1}{2}x^4 + x^3 - \frac{23}{2}x^2 - 12x\right]_{-4}^{-1/2} + \left[\frac{1}{2}x^4 + x^3 - \frac{23}{2}x^2 - 12x\right]_{-1/2}^{3}$$

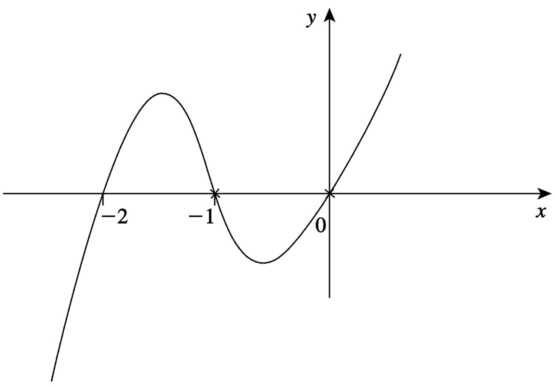
$$= \left[\frac{1}{2}\left(-\frac{1}{2}\right)^4 + \left(-\frac{1}{2}\right)^3 - \frac{23}{2}\left(-\frac{1}{2}\right)^2 - 12\left(-\frac{1}{2}\right)\right] - \left[\frac{1}{2}(-4)^4 + (-4)^3 - \frac{23}{2}(-4)^2 - 12(-4)\right]$$

$$+\left|\left(\frac{1}{2}(3)^4 + 3^3 - \frac{23}{2}(3)^2 - 12(3)\right) - \left(\frac{1}{2}\left(-\frac{1}{2}\right)^4 + \left(-\frac{1}{2}\right)^3 - \frac{23}{2}\left(-\frac{1}{2}\right)^2 - 12\left(-\frac{1}{2}\right)\right)\right|$$

$$= \left(\frac{97}{32} - (-72)\right) + \left|-72 - \frac{97}{32}\right|$$

$$= 150.0625$$

15



$$\text{Area} = \int_{-2}^{-1} x^3 + 3x^2 + 2x\ \ dx + \left|\int_{-1}^{0} x^3 + 3x^2 + 2x\,dx\right|$$

$$= \left[\frac{1}{4}x^4 + x^3 + x^2\right]_{-2}^{-1} + \left|\left[\frac{1}{4}x^4 + x^3 + x^2\right]_{-1}^{0}\right|$$

$$= \left[\frac{1}{4}(-1)^4 + (-1)^3 + (-1)^2\right] - \left[\frac{1}{4}(-2)^4 + (-2)^3 + (-2)^2\right]$$

$$+\left|0 - \left[\left(\frac{1}{4}\right)(-1)^4 + (-1)^3 + (-1)^2\right]\right|$$

$$= \left(\frac{1}{4} - 1 + 1\right) - (4 - 8 + 4) + \left|\left[-\frac{1}{4}\right]\right|$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Exercise 16 B

1 $V = \pi\int_a^b y^2\ dx$

$$= \pi\int_0^1 \left(\frac{1}{\sqrt{3x+1}}\right)^2 dx$$

$$= \pi\int_0^1 \frac{1}{3x+1} dx$$

$$= \left[\frac{\pi}{3} \ln|3x+1|\right]_0^1$$

$$= \frac{\pi}{3} \ln 4$$

2 (a) $(1 - \sqrt{2}\sin x)^2 = 1 - 2\sqrt{2}\sin x + 2\sin^2 x$

$$= 1 - 2\sqrt{2}\sin x + 2\left[\frac{1 - \cos 2x}{2}\right]$$

$$= 2 - 2\sqrt{2}\sin x - \cos 2x$$

(b) $V = \pi\int_0^{\pi/4} (1 - \sqrt{2}\sin x)^2\ dx$

$$= \pi\int_0^{\pi/4} 2 - 2\sqrt{2}\sin x - \cos 2x\ dx$$

$$= \pi\left[2x + 2\sqrt{2}\cos x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \pi\left[\left(\frac{2\pi}{4} + 2\sqrt{2}\cos\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - (2\sqrt{2}\cos 0)\right]$$

$$= \pi\left[\frac{\pi}{2} + 2 - \frac{1}{2} - 2\sqrt{2}\right]$$

$$= \pi\left[\frac{\pi}{2} - 2\sqrt{2} + \frac{3}{2}\right]$$

3 $y = 3x - x^2$

$V = \pi\int y^2\ dx$

$V = \pi\int_1^2 (3x - x^2)^2\ dx$

$$= \pi\int_1^2 9x^2 - 6x^3 + x^4\ dx$$

$$= \pi\left[3x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5\right]_1^2$$

$$= \pi\left[\left(24 - 24 + \frac{32}{5}\right) - \left(3 - \frac{3}{2} + \frac{1}{5}\right)\right]$$

$$= \pi\left[\frac{32}{5} - \frac{3}{2} - \frac{1}{5}\right]$$

$$= \frac{47}{10}\pi$$

4 $$V = \pi\int_0^{\pi/4} \sin^2 x \, dx$$

$$= \frac{\pi}{2}\int_0^{\pi/4} 1 - \cos 2x \, dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

5 $$V = \pi\int_0^{3\pi/2} y^2 \, dx$$

$$= \pi\int_0^{3\pi/2} (1 + \sin x)^2 \, dx$$

$$= \pi\int_0^{3\pi/2} 1 + 2\sin x + \sin^2 x \, dx$$

$$= \pi\int_0^{3\pi/2} 1 + 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$

$$= \pi\int_0^{3\pi/2} \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x \, dx$$

$$= \pi\left[\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x\right]_0^{3\pi/2}$$

$$= \pi\left[\left(\frac{9\pi}{4} - 2\cos\frac{3\pi}{2} - \frac{1}{4}\sin 3\pi\right) - (-2)\right]$$

$$= \pi\left[\frac{9\pi}{4} + 2\right]$$

6 $$V = \pi\int_0^{\pi/4} \sin^2 x \, dx + \pi\int_{\pi/4}^{\pi/2} \cos^2 x \, dx$$

$$= \pi\int_0^{\pi/4} \frac{1 - \cos 2x}{2} \, dx + \pi\int_{\pi/4}^{\pi/2} \frac{1 + \cos 2x}{2} \, dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4} + \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] + \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$= \frac{\pi}{2}\left[\frac{\pi}{2} - 1\right]$

7 Gradient of AB $= \frac{\frac{1}{3} - 3}{3 - 1} = \frac{-\frac{8}{3}}{2} = -\frac{4}{3}$

Eq of AB:

$y - 3 = -\frac{4}{3}(x - 1)$

$y = -\frac{4}{3}x + \frac{4}{3} + 3$

$y = -\frac{4}{3}x + \frac{13}{3}$

Volume under the curve $= \pi \int_1^3 \frac{9}{x^4}\,dx$

$= \pi\left[\frac{-3}{x^3}\right]_1^3$

$= \pi\left[-\frac{1}{9} + 3\right] = \frac{26}{9}\pi$

Volume under the line $= \pi \int_a^b y^2\,dx$

$= \pi \int_1^3 \left(\frac{-4}{3}x + \frac{13}{3}\right)^2 dx$

$= \frac{\pi}{9}\int_1^3 (13 - 4x)^2\,dx$

$= \frac{\pi}{9}\left[\frac{-(13-4x)^3}{12}\right]_1^3$

$= \frac{\pi}{9}\left[-\frac{1}{12} + \frac{9^3}{12}\right]$

$= \frac{182}{27}\pi$

Required volume $= \frac{182}{27}\pi - \frac{26}{9}\pi$

$= \frac{104}{27}\pi$

8 (a) Gradient of AB $= \frac{\frac{1}{5} - 5}{5 - 1} = \frac{-\frac{24}{5}}{4} = -\frac{6}{5}$

Equation of AB = $y - 5 = -\frac{6}{5}(x - 1)$

$y = \frac{-6}{5}x + \frac{6}{5} + 5$

$y = \frac{-6}{5}x + \frac{31}{5}$

(b) Volume $= \pi \int_1^5 \left(-\frac{6}{5}x + \frac{31}{5}\right)^2 dx$

$= \frac{\pi}{25}\int_1^5 (-6x+31)^2\, dx$

$= \frac{\pi}{25}\left[\frac{(-6x+31)^3}{-18}\right]_1^5 = \frac{\pi}{25}\left[-\frac{1}{18} + \frac{25^3}{18}\right]$

$= 34.72\,\pi$

Volume curve $= \pi \int_1^5 \frac{25}{x^4}\, dx = \frac{-25\pi}{3}\left[\frac{1}{x^3}\right]_1^5$

$= \frac{-25\pi}{3}\left[\frac{1}{125} - 1\right]$

$= 8.27\,\pi$

Required volume = $34.72\pi - 8.27\pi$

$= 26\pi$

9 $y = -\cos\left(2x + \frac{\pi}{6}\right) = 0$

$2x + \frac{\pi}{6} = \frac{\pi}{2}$

$x = \frac{\pi}{6}$

volume below x-axis $= \pi \int_0^{\pi/6} \cos^2\left(2x + \frac{\pi}{6}\right) dx$

$= \frac{\pi}{2}\int_0^{\pi/6} 1 + \cos\left(4x + \frac{\pi}{3}\right) dx$

$= \frac{\pi}{2}\left[x + \frac{1}{4}\sin\left(4x + \frac{\pi}{3}\right)\right]_0^{\pi/6}$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{6} + \frac{1}{4}\sin\pi\right) - \frac{1}{4}\sin\left(\frac{\pi}{3}\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right] = 0.4824$

volume above x-axis $= \frac{\pi}{2}\left[x + \frac{1}{4}\sin\left(4x + \frac{\pi}{3}\right)\right]_{\pi/6}^{\pi/3}$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{3} + \frac{1}{4}\sin\frac{5\pi}{3}\right) - \left(\frac{\pi}{6} + \frac{1}{4}\sin\pi\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{8} - \frac{\pi}{6}\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right] = 0.4828$

Required volume = 0.965

10 $V = \pi \int_0^{\pi/4} \tan^2 x\, dx$

$$= \pi \int_0^{\pi/4} \sec^2 x - 1\,dx$$

$$= \pi\,[\tan x - x]_0^{\pi/4}$$

$$= \pi\left[\tan\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= \pi\left[1 - \frac{\pi}{4}\right]$$

11 $V = \pi \int_0^{3\pi/8} 4\sin^2\left(2x + \frac{\pi}{4}\right) dx$

$$= 2\pi \int_0^{3\pi/8} 1 - \cos\left(4x + \frac{\pi}{2}\right)\Big]\,dx$$

$$= 2\pi\left[x - \frac{1}{4}\sin\left(4x + \frac{\pi}{2}\right)\right]_0^{3\pi/8}$$

$$= 2\pi\left[\left\{\frac{3\pi}{8} - \frac{1}{4}\sin(2\pi)\right\} - \left\{\frac{1}{4}\sin\frac{\pi}{2}\right\}\right]$$

$$= 2\pi\left[\frac{3\pi}{8} - \frac{1}{4}\right]$$

12 $y = 2\sin x + 4\cos x$

$$V = \pi \int_0^a (2\sin x + 4\cos x)^2\,dx$$

$$= \pi \int_0^a 4\sin^2 x + 16\sin x\cos x + 16\cos^2 x\,dx$$

$$= \pi \int_0^a 4 + 8\sin 2x + 12\left[\frac{1 + \cos 2x}{2}\right] dx$$

$$= \pi \int_0^a 10 + 8\sin 2x + 6\cos 2x\,dx$$

$$= \pi\,[10x - 4\cos 2x + 3\sin 2x]_0^a$$

$$= \pi\,[10a - 4\cos 2a + 3\sin 2a + 4]$$

13 $V = \pi \int_1^8 y^{2/3}dx$

$$= \pi\left[\frac{3y^{5/3}}{5}\right]_1^8$$

$$= \frac{3\pi}{5}[8^{5/3} - 1]$$

$$= \frac{3\pi}{5}[32 - 1] = \frac{93}{5}\pi$$

14 $x = \frac{2}{y}$

$$x^2 = \frac{4}{y^2}$$

$$V = \pi \int_2^5 \frac{4}{y^2}\,dy$$

$= \pi \left[-\frac{4}{y} \right]_2^5$

$= \pi \left[-\frac{4}{5} + 2 \right]$

$= \frac{6}{5}\pi$

Review exercise 16

1 $\int_0^{\pi/4} \sin x \, dx = [-\cos x]_0^{\pi/4} = -\cos\frac{\pi}{4} + \cos 0$

$= -\frac{\sqrt{2}}{2} + 1$

$\int_0^{\pi/4} \cos x \, dx = [\sin x]_0^{\pi/4} = \sin\frac{\pi}{4} - \sin 0$

$= \frac{\sqrt{2}}{2}$

Required area $= \frac{\sqrt{2}}{2} - \left(1 - \frac{\sqrt{2}}{2}\right)$

$= (\sqrt{2} - 1)$ Square units

2 $y = x^2 - 2x,\ \ y = 6x - x^2$

$x^2 - 2x = 6x - x^2$

$2x^2 - 8x = 0$

$2x(x - 4) = 0$

$x = 0, 4$

For $y = x^2 - 2x$

if $y = 0,\ \ x = 0, 2$

so curve is below x-axis from $x = 0$ to $x = 2$, need to find the area in two sections,

$0 < x < 2,\ 2 < x < 4$

Area above $y = x^2 - 2x = \int_0^2 (x^2 - 2x)\, dx$

$= \left| \left[\frac{1}{3}x^3 - x^2 \right]_0^2 \right|$

$= \left| \frac{8}{3} - 4 \right| = \frac{4}{3}$

Area under $y = x^2 - 2x = \int_2^4 (x^2 - 2x)\, dx$

$= \left[\frac{1}{3}x^3 - x^2 \right]_2^4$

$= \frac{64}{3} - 16 - \frac{8}{3} + 4 = \frac{20}{3}$

Area under $y = 6x - x^2 = \int_0^4 (6x - x^2)\, dx$

$= \left[3x^2 - \frac{1}{3}x^3\right]_0^4$

$= 48 - \frac{64}{3} = \frac{80}{3}$

Required area = area under $y = 6x - x^2$ – area under $y = x^2 - 2x$ + area above $y = x^2 - 2x$

$= \frac{80}{3} - \frac{20}{3} + \frac{4}{3} = \frac{56}{3}$ Square units

3 $y = x^3 - 6x^2 + 8x$.

$\frac{dy}{dx} = 3x^2 - 12x + 8$

$\frac{d^2y}{dx^2} = 6x - 12$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x + 8 = 0$

$x = \frac{12 \pm \sqrt{48}}{6}$

$= \frac{12 \pm 4\sqrt{3}}{6}$

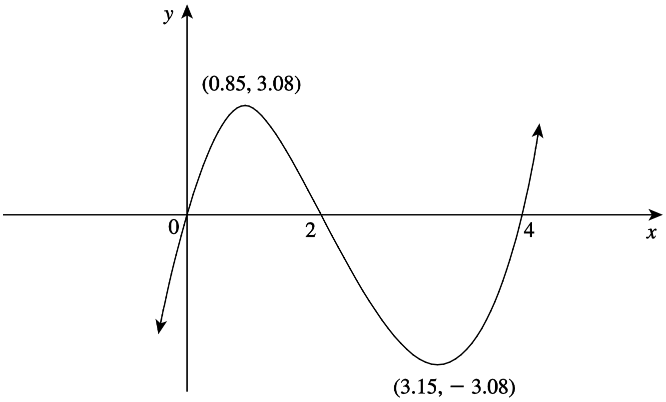
$x = 3.15, 0.85$

when $x = 0.85$, $\frac{d^2y}{dx^2} = 6(0.85) - 12 < 0 \Rightarrow$ max point

$x = 3.15$, $\frac{d^2y}{dx^2} = 6(3.15) - 12 > 0 \Rightarrow$ min point

$x = 0.85$, $y = 3.08$

$x = 3.15$, $y = -3.08$



Area $= \int_0^2 x^3 - 6x^2 + 8x\,dx + \left|\int_2^4 (x^3 - 6x^2 + 8x)\,dx\right|$

$= \left[\frac{1}{4}x^4 - 2x^3 + 4x^2\right]_0^2 + \left|\left[\frac{1}{4}x^4 - 2x^3 + 4x^2\right]_2^4\right|$

$= (4 - 16 + 16) + |(64 - 128 + 64) - (4 - 16 + 16)|$

$= 4 + |-4|$

= 8 Square units

4 $y = x^2 - 4$

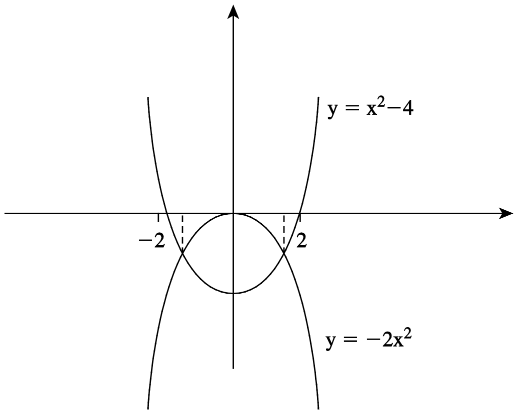
$y = -2x^2$

$-2x^2 = x^2 - 4$

$3x^2 = 4$

$x = \pm\sqrt{\frac{4}{3}}$

$x = \pm\frac{2\sqrt{3}}{3}$



For $y = x^2 - 4$ Area $= \int_{-\frac{2\sqrt{3}}{3}}^{\frac{2\sqrt{3}}{3}} x^2 - 4\,dx$

$= \left[\frac{1}{3}x^3 - 4x\right]_{-\frac{2\sqrt{3}}{3}}^{\frac{2\sqrt{3}}{3}}$

$= \frac{-64}{27}\sqrt{3} - \left[\frac{64\sqrt{3}}{27}\right]$

$= -\frac{128}{27}\sqrt{3}$

For $y = -2x^2$

Area $= \int_{-\frac{2\sqrt{3}}{3}}^{\frac{2\sqrt{3}}{3}} -2x^2\,dx$

$= \left[\frac{-2}{3}x^3\right]_{-\frac{2\sqrt{3}}{3}}^{\frac{2\sqrt{3}}{3}}$

$= \frac{-32\sqrt{3}}{27}$

$\therefore$ Required area $= \frac{128}{27}\sqrt{3} - \frac{32}{27}\sqrt{3}$

= 6.16 Square units

5 (a) Gradient of AB $= \frac{2-0}{3-1} = 1$

Eq of AB: $y - 0 = x - 1$

$y = x - 1$

(b) $y = 0 \Rightarrow x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

curve is above the x-axis from x = 2

Volume under the curve from x = 2:

$$\pi \int_2^3 (x^2 - 3x + 2)^2 \, dx = \pi \int_2^3 (x^4 - 6x^3 + 13x^2 - 12x + 4)\, dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{3}{2}x^4 + \frac{13}{3}x^3 - 6x^2 + 4x\right]_2^3 = \pi\left[\frac{21}{10} - \frac{16}{15}\right] = \frac{31}{30}\pi$$

Volume under the line: $\pi \int_1^3 (x-1)^2 \, dx = \pi\left[\frac{1}{3}(x-1)^3\right]_1^3$

$$= \frac{8}{3}\pi$$

Required volume $\frac{8}{3}\pi - \frac{31}{30}\pi = \frac{49}{30}\pi$

6 (a) $V = \pi \int_a^b y^2 \, dx$

$$V = \pi \int_1^2 \left(\frac{1}{4x-1}\right)^2 dx$$

$$= \pi \int_1^2 (4x-1)^{-2}\, dx$$

$$= \pi\left[-\frac{1}{4}(4x-1)^{-1}\right]_1^2$$

$$= \pi\left[-\frac{1}{28} + \frac{1}{12}\right]$$

$$= \frac{1}{21}\pi$$

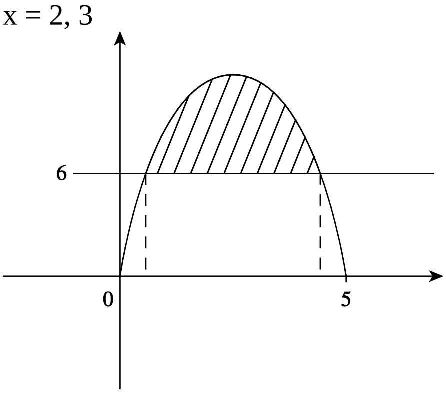
(b) $y = 6$

$y = 5x - x^2$

$5x - x^2 = 6 \Rightarrow x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2, 3$



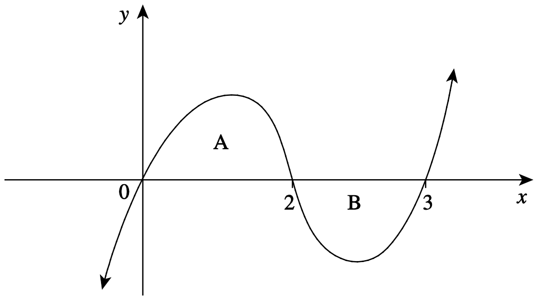
Required area $= \int_2^3 (5x - x^2)\, dx - \int_2^3 6\, dx$

$$= \left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_2^3 - [6x]_2^3$$

$$= \left(\frac{45}{2} - 9\right) - \left(10 - \frac{8}{3}\right) - [18 - 12]$$

$$= \frac{1}{6}.$$

7



Area of region A $= \int_0^2 x^3 - 5x^2 + 6x \, dx$

$$= \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \left(4 - \frac{40}{3} + 12\right) = \frac{8}{3}$$

Area of region B =

$$\left|\int_2^3 \left(x^3 - 5x^2 + 6x\right) dx\right| = \left|\left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_2^3\right| = \left|\left(\frac{81}{4} - 45 + 27\right) - \left(\frac{8}{3}\right)\right| = \frac{5}{12}$$

Ratio of $A:B \ \frac{8/3}{5/12} = 32:5$

8 (a) $y = x^3 + 4x^2 + 3x$

$= x\,(x^2 + 4x + 3)$

$= x\,(x + 1)\,(x + 3)$

$y = 0,\ x = 0, -1, -3$

$\frac{dy}{dx} = 3x^2 + 8x + 3$

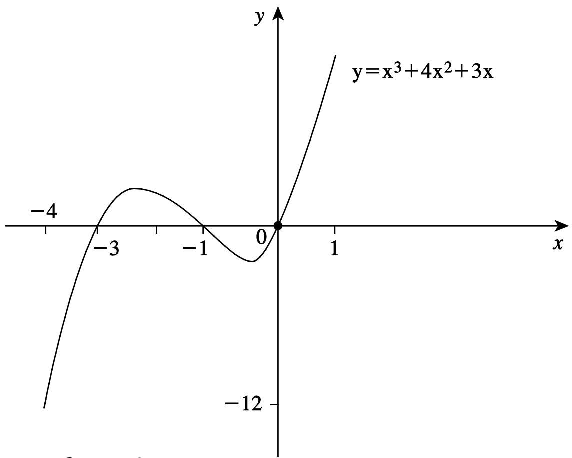
$\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 8x + 3 = 0$

$x = \frac{-8 \pm \sqrt{28}}{6}$

$= -0.45, -2.22$

$x = -0.45,\ \frac{d^2y}{dx^2} = 6x + 8 = 6(-0.45) + 8 > 0$ min point

$x = -2.22,\ \frac{d^2y}{dx^2} = 6(-2.22) + 8 < 0$ max point



(b) $y = x^3 + 4x^2 + 3x$

When $y = 0 \Rightarrow x^3 + 4x^2 + 3x = 0$

$x(x^2 + 4x + 3) = 0$
$x(x + 3)(x + 1) = 0$
$x = 0, -1, -3$

Now $\frac{dy}{dx} = 3x^2 + 8x + 3$

When $x = 0$, $\frac{dy}{dx} = 3$

$x = -1$, $\frac{dy}{dx} = 3(-1)^2 + 8(-1) + 3 = -2$

$x = -3$, $\frac{dy}{dx} = 3(-3)^2 + 8(-3) + 3 = 27 - 24 + 3 = 6$

$\therefore$ When $x = 0$, $\frac{dy}{dx} = 3$

$x = -1$, $\frac{dy}{dx} = -2$

$x = -3$, $\frac{dy}{dx} = 6$

(c) Area from $x = -1$ to $x = -3$ = $\int_{-3}^{-1} (x^3 + 4x^2 + 3x)\,dx$

$= \left[\frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_{-3}^{-1}$

$= \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - \left(\frac{81}{4} - 36 + \frac{27}{2}\right)$

$= \frac{8}{3}$ Square units

9 $V = \pi \int_a^b y^2\,dx$

$\Rightarrow V = \pi \int_1^4 \left(\frac{6}{x+2}\right)^2 dx$

$= \pi \int_1^4 36(x+2)^{-2}\,dx$

$= 36\pi \left[-\frac{1}{x+2}\right]_1^4$

$= 36\pi \left[-\frac{1}{6} + \frac{1}{3}\right]$

$= 6\pi$.

10 Rotation about the y-axis:

$V = \pi \int_c^d x^2\,dy$

Since $x - y^2 - 4 = 0$
$\Rightarrow x = y^2 + 4$
$x^2 = (y^2 + 4)^2 = y^4 + 8y^2 + 16$

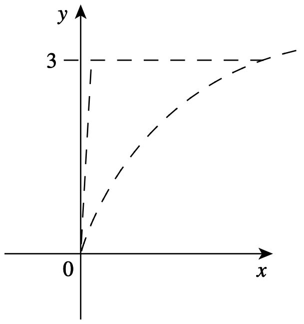
$V = \pi \int_0^3 (y^4 + 8y^2 + 16)\,dy$

$$= \pi\left[\frac{1}{5}y^5 + \frac{8}{3}y^3 + 16y\right]_0^3$$

$$= \pi\left[\frac{243}{5} + 72 + 48\right]$$

$$= 168\frac{3}{5}\pi$$

11



$$V = \pi\int_c^d x^2\, dy$$

$$y = \sqrt{4x}$$

$$\frac{y^2}{4} = x \Rightarrow x^2 = \frac{1}{16}y^4$$

$$V = \pi\int_0^3 \frac{1}{16}y^4\, dy$$

$$= \pi\left[\frac{1}{80}y^5\right]_0^3$$

$$= \frac{243}{80}\pi$$

12 $V = \pi\int_2^3 (9 - x^2)^2\, dx$

$$= \pi\int_2^3 (81 - 18x^2 + x^4)\, dx$$

$$= \pi\left[81x - 6x^3 + \frac{1}{5}x^5\right]_2^3$$

$$= \pi\left[81(3) - 6(3)^3 + \frac{1}{5}(3)^5 - \left[81(2) - 6(2)^3 + \frac{1}{5}(2)^5\right]\right]$$

$$= \frac{46}{5}\pi$$

13 Equation of the line OP:

$y = 3x$

Required volume $= \pi\int_0^1 9x\, dx - \pi\int_0^1 9x^2\, dx$

$$= \frac{9\pi}{2}[x^2]_0^1 - 3\pi[x^3]_0^1$$

$= \frac{9}{2}\pi - 3\pi$

$= \frac{3}{2}\pi$

14 (a) Gradient of AB $= \dfrac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \dfrac{2\sqrt{2}}{\pi}$

Equation $y = \dfrac{2\sqrt{2}}{\pi}x$

(b) Shaded volume $= \pi\int_0^{\pi/4} \sin^2 x\, dx - \pi\int_0^{\pi/4}\left(\frac{2\sqrt{2}}{\pi}x\right)^2 dx$

$= \frac{\pi}{2}\int_0^{\pi/4} 1 - \cos 2x\, dx - \frac{8}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi/4}$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4} - \frac{8}{\pi}\left(\frac{\pi^3}{64}\right)\left(\frac{1}{3}\right)$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right] - \frac{\pi^2}{24}$

$= \frac{\pi^2}{8} - \frac{\pi}{4} - \frac{\pi^2}{24} = \frac{\pi^2}{12} - \frac{\pi}{4}$

15 For $y = 9 - x^2$

Area $= \int_0^3 (9 - x^2)\, dx$

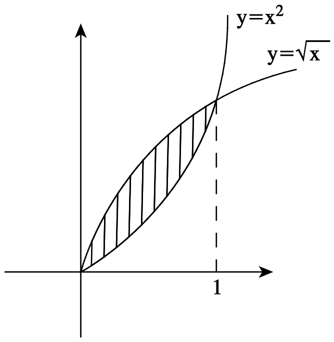
$= \left[9x - \frac{1}{3}x^3\right]_0^3 = 27 - 9 = 18$

For $y = x^2 + 1$:

Area $= \int_0^3 (x^2 + 1)\, dx = \left[\frac{1}{3}x^3 + x\right]_0^3 = 9 + 3 = 12$

$\therefore$ Required area = 18 – 12 = 6

16



$x^2 = \sqrt{x}$

$\Rightarrow x^4 = x$

$x = 0, 1$

(a) $V = \pi\int_0^1 (\sqrt{x})^2\, dx - \pi\int_0^1 (x^2)^2\, dx$

$= \pi \int_0^1 x\,dx - \pi \int_0^1 x^4\,dx$

$= \frac{\pi}{2}[x^2]_0^1 - \frac{\pi}{5}[x^5]_0^1$

$= \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$

(b) when x = 1, y = 1

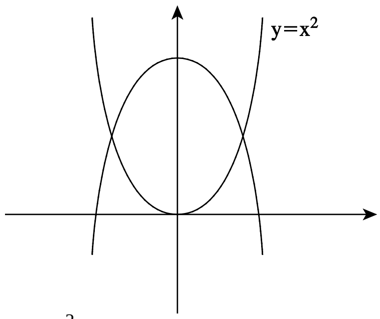
$y = x^2, \quad y = \sqrt{x} \Rightarrow y^4 = x^2$

$V = \pi \int_0^1 y\,dy - \pi \int_0^1 y^4\,dy$

$= \frac{\pi}{2}[y^2]_0^1 - \frac{\pi}{5}[y^5]_0^1$

$= \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$

17 (a)



(b) $y = x^2$

$y = 18 - x^2$

$18 - x^2 = x^2$

$2x^2 = 18$

$x^2 = 9 \Rightarrow x = \pm 3$

when x = 3, y = 9

x = –3, y = 9

Points of intersections (3, 9) and (–3, 9)

(c) $\int_{-3}^{3} 18 - x^2\,dx = \left[18x - \frac{1}{3}x^3\right]_{-3}^{3}$

$= (54 - 9) - (-54 + 9)$

$= 90$

$\int_{-3}^{3} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-3}^{3} = 9 + 9 = 18$

Required area = 90 – 18 = 72

18 (a) sin 2x = cos x

$\Rightarrow 2\sin x\cos x - \cos x = 0$

$\Rightarrow \cos x\,(2\sin x - 1) = 0$

$\cos x = 0, \ \sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \ x = \frac{\pi}{6}$

Points of intersections: $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{\pi}{2}, 0\right)$

(b) $\int_{\frac{\pi}{6}}^{\pi/2} \sin 2x\, dx = \left[\frac{-1}{2}\cos 2x\right]_{\pi/6}^{\pi/2}$

$= -\frac{1}{2}\cos\pi + \frac{1}{2}\cos\frac{\pi}{3}$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_{\pi/6}^{\pi/2} = \sin\frac{\pi}{2} - \sin\frac{\pi}{6} = 1 - \frac{1}{2} = \frac{1}{2}$

Required area $= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$