Chapter 15 Integration

Exercise 15A

1 $\int x^9\,dx = \frac{1}{10}x^{10} + c$

2 $\int 5x^6\,dx = \frac{5}{7}x^7 + c$

3 $\int \frac{1}{2}x^{1/2}\,dx = \frac{1}{3}x^{3/2} + c$

4 $\int 8\,x^6\,dx = \frac{8}{7}x^7 + c$

5 $\int \frac{x+1}{x^3}\,dx = \int \frac{1}{x^2} + \frac{1}{x^3}dx$

$= \int x^{-2} + x^{-3}\;dx$

$= \frac{-1}{x} - \frac{1}{2x^2} + c$

6 $\int (4x+1)(x^2+2)\,dx = \int 4x^3 + x^2 + 8x + 2\,dx$

$= x^4 + \frac{1}{3}x^3 + 4x^2 + 2x + c$

7 $\int \frac{\sqrt{x}-2}{\sqrt{x}}\,dx = \int 1 - 2x^{-1/2}\,dx = x - 4x^{1/2} + c$

$= x - 4\sqrt{x} + c$

8 $\int \frac{x^4 - 6x^2}{x^6}\,dx = \int x^{-2} - 6x^{-4}\,dx = -\frac{1}{x} + \frac{2}{x^3} + c$

9 $\int (2x^2+1)^2\,dx = \int 4x^4 + 4x^2 + 1\,dx = \frac{4}{5}x^5 + \frac{4}{3}x^3 + x + c$

10 $\int \frac{(\sqrt{x}+1)^2}{\sqrt{x}}\,dx = \int \frac{x + 2\sqrt{x} + 1}{\sqrt{x}}\,dx$

$= \int x^{1/2} + 2 + x^{-1/2}\;dx$

$= \frac{2}{3}x^{3/2} + 2x + 2x^{1/2} + c$

$= \frac{2}{3}x\sqrt{x} + 2x + 2\sqrt{x}\; + c$

11 $\int (x - 3\sqrt{x})^2\,dx = \int x^2 - 6x\sqrt{x} + 9x\,dx.$

$= \int x^2 - 6x^{3/2} + 9x\,dx$

$= \frac{1}{3}x^3 - \frac{12}{5}x^{5/2} + \frac{9}{2}x^2 + c$

12 $\int \frac{x^5 + 6x^2}{x^7}\,dx = \int x^{-2} + 6x^{-5}\,dx$

$= -\frac{1}{x} - \frac{6}{4}x^{-4} + c$

$= -\frac{1}{x} - \frac{3}{2x^4} + c$

13 $\int \frac{x^3+4}{x^3}\,dx = \int 1 + 4x^{-3}dx$

$= x - \frac{4}{2}x^{-2} + c$

$= x - \frac{2}{x^2} + c$

14 $\int 4x^3 - \frac{5}{x^2}dx = \int 4x^3 - 5x^{-2}\,dx$

$= x^4 + \frac{5}{x} + c$

15 $\int (x^2+2)^2\,dx = \int x^4 + 4x^2 + 4\,dx$

$= \frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x + c$

16 $\int 4x^5\,dx = \frac{4}{6}x^6 + c = \frac{2}{3}x^6 + c$

17 $\int 7x^3 - 3x^{-5}\,dx = \frac{7}{4}x^4 + \frac{3}{4}x^{-4} + c$

$= \frac{7}{4}x^4 + \frac{3}{4x^4} + c$

18 $\int (4+2x)^3\,dx$

$= \int 16 + 3(4)^2(2x) + 3(4)(2x)^2 + (2x)^3\,dx$

$= \int 16 + 96x + 48x^2 + 8x^3\,dx$

$= 16x + 48x^2 + 16x^3 + 2x^4 + c$

19 $\int (x-1)^3\,dx$

$= \int x^3 - 3x^2 + 3x - 1\,dx$

$= \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - x + c$

20 $\int \left(\frac{x+1}{x^2}\right)^2 dx$

$= \int \frac{x^2+2x+1}{x^4}\,dx$

$= \int \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}\,dx$

$= \int x^{-2} + 2x^{-3} + x^{-4}\,dx$

$= -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{3x^3} + c$

21 $(1 - 3x + x^2)^2 = (1 - 3x + x^2)(1 - 3x + x^2)$

$= 1 - 3x + x^2 - 3x + 9x^2 - 3x^3 + x^2 - 3x^3 + x^4$

$= 1 - 6x + 11x^2 - 6x^3 + x^4$

$\int (1-3x+x^2)^2\,dx = \int 1 - 6x + 11x^2 - 6x^3 + x^4\,dx$

$= x - 3x^2 + \frac{11}{3}x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5 + c$

22 $\int \frac{4}{x^2} + 6x^3 + 10\sqrt{x}\, dx$

$= \int 4x^{-2} + 6x^3 + 10x^{1/2}\, dx$

$= -\frac{4}{x} + \frac{3}{2}x^4 + \frac{20}{3}x^{3/2} + c$

23 $\int 4x^7 + 3x^3 - \frac{6}{x^3}\, dx$

$= \int 4x^7 + 3x^3 - 6x^{-3}\, dx$

$= \frac{1}{2}x^8 + \frac{3}{4}x^4 + \frac{3}{x^2} + c$

24 $\int \frac{x^2 + 4x}{\sqrt{x}}\, dx$

$= \int x^{3/2} + 4x^{1/2}\, dx$

$= \frac{2}{5}x^{5/2} + \frac{8}{3}x^{3/2} + c$

25 $\int \frac{3}{t^3} + \frac{2}{t^2}\, dt = \int 3t^{-3} + 2t^{-2}\, dt$

$= \frac{-3}{2}t^{-2} - \frac{2}{1}t^{-1} + c$

$= \frac{-3}{2t^2} - \frac{2}{t} + c$

26 $\frac{dy}{dx} = 4x^3 - 8x + 2$

$y = \int 4x^3 - 8x + 2\, dx$

$y = x^4 - 4x^2 + 2x + c$

$x = 1, y = 0 \Rightarrow 0 = 1 - 4 + 2 + c$

$c = 1$

$y = x^4 - 4x^2 + 2x + 1$

27 $\frac{dy}{dx} = 4x + 3$

$y = \int 4x + 3\, dx$

$y = 2x^2 + 3x + c$

$x = 2, y = 1 \Rightarrow 1 = 8 + 6 + c$

$c = -13$

$y = 2x^2 + 3x - 13$

28 $\frac{dy}{dt} = \frac{4t - 3}{t^3}$

$= \frac{4}{t^2} - \frac{3}{t^3}$

$y = \int 4t^{-2} - 3t^{-3}\, dt$

$y = \frac{-4}{t} + \frac{3}{2t^2} + c$

$t = 2, y = 1 \Rightarrow 1 = \frac{-4}{2} + \frac{3}{2(2)^2} + c$

$1 = -2 + \frac{3}{8} + c$

$c = 3 - \frac{3}{8} = \frac{21}{8}$

$y = \frac{-4}{t} + \frac{3}{2t^2} + \frac{21}{8}$

29 $\frac{dy}{dx} = x^2 - 6 + \frac{1}{2}x$

$y = \int x^2 - 6 + \frac{1}{2}x\, dx$

$y = \frac{1}{3}x^3 - 6x + \frac{1}{4}x^2 + c$

$x = 1, y = 1 \Rightarrow 1 = \frac{1}{3} - 6 + \frac{1}{4} + c$

$c = 7 - \frac{1}{3} - \frac{1}{4}$

$= \frac{77}{12}$

$y = \frac{1}{3}x^3 + \frac{1}{4}x^2 - 6x + \frac{77}{12}$

30 $\frac{du}{dt} = 4t^2 - \sqrt{t}$

$u = \int 4t^2 - t^{1/2}\, dt$

$u = \frac{4}{3}t^3 - \frac{2}{3}t^{3/2} + c$

$u = 1, t = 1 \Rightarrow 1 = \frac{4}{3} - \frac{2}{3} + c$

$\frac{1}{3} = c$

$u = \frac{4}{3}t^3 - \frac{2}{3}t^{3/2} + \frac{1}{3}$

Try these 15.1

(a) (i) $\int (4x+2)^8\, dx = \frac{1}{4}\left[\frac{(4x+2)^9}{9}\right] + c = \frac{1}{36}(4x+2)^9 + c$

(ii) $\int \sqrt{3x+5}\, dx = \int (3x+5)^{\frac{1}{2}}\, dx = \frac{1}{3}\frac{(3x+5)^{3/2}}{3/2} + c = \frac{2}{9}(3x+5)^{3/2} + c$

(iii) $\int \sqrt{2x+1}\, dx = \int (2x+1)^{\frac{1}{2}}\, dx = \frac{1}{2}\frac{(2x+1)^{3/2}}{3/2} + c = \frac{1}{3}(2x+1)^{3/2} + c$

(b) $\frac{dy}{dx} = (3x-1)^6$

$$\Rightarrow y = \int (3x-1)^6\, dx$$

$$y = \frac{1}{3}\frac{(3x-1)^7}{7} + c$$

$$y = \frac{1}{21}(3x-1)^7 + c$$

When x = 1, y = 1 $\Rightarrow 1 = \frac{2^7}{21} + c$

$$c = 1 - \frac{128}{21}$$

$$= \frac{-107}{21}$$

Hence $y = \frac{1}{21}(3x-1)^7 - \frac{107}{21}$

Try these 15.2

(a) $\int 4\sin\left(x + \frac{\pi}{2}\right) dx = -4\cos\left(x + \frac{\pi}{2}\right) + c$

(b) $\int \cos\left(\frac{\pi}{2} - 3x\right) dx = -\frac{1}{3}\sin\left(\frac{\pi}{2} - 3x\right) + c$

(c) $\int 4\tan\left(3x + \frac{\pi}{2}\right) dx = \frac{4}{3}\ln\left|\sec\left(3x + \frac{\pi}{2}\right)\right| + c$

(d) $\int 3\cot(5x)\, dx = \frac{3}{5}\ln|\sin 5x| + c$

Exercise 15 B

1 (a) $\int \sin 6x\, dx = -\frac{1}{6}\cos 6x + c$

(b) $\int \cos 3x\, dx = \frac{1}{3}\sin 3x + c$

(c) $\int 4\cos 2x = 2\sin 2x + c$

(d) $\int 2\sec^2 3x\, dx = \frac{2}{3}\tan 3x + c$

(e) $\int 2\cos 6x - \sin 4x\, dx = \frac{1}{3}\sin 6x + \frac{1}{4}\cos 4x + c$

(f) $\int \cos 4x + \frac{1}{\cos^2 3x}\, dx = \int \cos 4x + \sec^2 3x\, dx$

$$= \frac{1}{4}\sin 4x + \frac{1}{3}\tan 3x + c$$

(g) $\int \sin 4x + \cos 5x + 3\cos x\, dx = -\frac{1}{4}\cos 4x + \frac{1}{5}\sin 5x + 3\sin x + c$

(h) $\int x^2 + 6\tan 2x\, dx = \frac{1}{3}x^3 + 3\ln|\sec 2x| + c$

(i) $\int \frac{1}{x^3} - \sec^2(\sqrt{2}\,x)\,dx = \int x^{-3} - \sec^2(\sqrt{2}x)\,dx$

$= -\frac{1}{2x^2} - \frac{1}{\sqrt{2}}\tan(\sqrt{2}\,x) + c$

(j) $\int 4\sec^2(6x+9) = \frac{4}{6}\tan(6x+9) + c$

$= \frac{2}{3}\tan(6x+9) + c$

2 (a) $\int \sin(3x-1)\,dx = -\frac{1}{3}\cos(3x-1) + c$

(b) $\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + c$

(c) $\int \tan\left(\frac{\pi}{3} - 4x\right)dx = -\frac{1}{4}\ln\left|\sec\left(\frac{\pi}{3} - 4x\right)\right| + c$

(d) $\int \sec^2\left(x - \frac{3\pi}{4}\right)dx = \tan\left(x - \frac{3\pi}{4}\right) + c$

(e) $\int 7\cos(6x+9)\,dx = \frac{7}{6}\sin(6x+9) + c$

(f) $\int 6\sin(4x-6)\,dx = \frac{-3}{2}\cos(4x-6) + c$

(g) $\int 8\sec^2(8x-\pi)\,dx = \tan(8x-\pi) + c$

(h) $\int \cos qx\,dx = +\frac{1}{q}\sin qx + c$

(i) $\int \sin(px+\pi)\,dx = -\frac{1}{p}\cos(px+\pi) + c$

(j) $\int \tan(4-rx)\,dx = -\frac{1}{r}\ln|\sec(4-rx)| + c$

3 (a) $\int (3t+1)^6\,dt = \frac{1}{21}(3t+1)^7 + c$

(b) $\int (1-4t)^3\,dt = -\frac{1}{4}\left(\frac{1}{4}\right)(1-4t)^4 + c$

$= -\frac{1}{16}(1-4t)^4 + c$

(c) $\int (2t+7)^{-4} = \frac{1}{2}\frac{(2t+7)^{-3}}{-3} + c = \frac{-1}{6}(2t+7)^{-3} + c$

(d) $\int \sqrt{6t-1}\,dt = \int (6t-1)^{1/2}\,dt = \frac{1}{6} \times \frac{2}{3}(6t-1)^{3/2} + c$

$= \frac{1}{9}(6t-1)^{3/2} + c$

(e) $\int \frac{4}{\sqrt{2-3t}}\,dt = \int 4(2-3t)^{-1/2}\,dt$

$= \frac{4}{-3}\frac{(2-3t)^{1/2}}{1/2} + c$

$= \frac{-8}{3}(2-3t)^{1/2} + c$

(f) $\int \frac{4}{\sqrt{7-6t}}\,dt = \int 4(7-6t)^{-1/2}\,dt$

$= -\frac{4}{6}\frac{(7-6t)^{1/2}}{1/2} + c$

$= \frac{-4}{3}(7-6t)^{1/2} + c$

(g) $\int \frac{6}{7(3t-1)^3}\,dt = \frac{6}{7}\int (3t-1)^{-3}\,dt$

$= \frac{6}{7} \times \frac{1}{3}\frac{(3t-1)^{-2}}{-2} + c$

$= -\frac{1}{7}(3t-1)^{-2} + c$

(h) $\int \frac{2}{(4t-3)^5}\,dt = \int 2(4t-3)^{-5}\,dt$

$= \frac{\frac{2}{4}(4t-3)^{-4}}{-4} + c$

$= -\frac{1}{8}(4t-3)^{-4} + c$

$= \frac{-1}{8(4t-3)^4} + c$

(i) $\int 3(6t-4)^8\,dt = \frac{3}{6}\frac{(6t-4)^9}{9} + c$

$= \frac{1}{18}(6t-4)^9 + c$

4 $\frac{dy}{dx} = \frac{1}{(3x+2)^4}$

$y = \int (3x+2)^{-4}\,dx$

$y = \frac{1}{3}\frac{(3x+2)^{-3}}{-3} + c$

$y = \frac{-1}{9(3x+2)^3} + c$

$x = 0,\ y = 2 \Rightarrow\ 2 = \frac{-1}{9(8)} + c$

$c = 2 + \frac{1}{72}$

$= \frac{145}{72}$

$\therefore$ Equation of the curve is

$y = -\frac{1}{9(3x+2)^3} + \frac{145}{72}$

5 $\frac{dx}{dt} = \cos\left(2t - \frac{\pi}{4}\right)$

$x = \int \cos\left(2t - \frac{\pi}{4}\right) dt$

$x = +\frac{1}{2}\sin\left(2t - \frac{\pi}{4}\right) + c$

$x = 1,\ t = \frac{\pi}{4}$

$1 = \frac{1}{2}\sin\frac{\pi}{4} + c$

$c = 1 - \frac{\sqrt{2}}{4}$

$\therefore\ x = \frac{1}{2}\sin\left(2t - \frac{\pi}{4}\right) + 1 - \frac{\sqrt{2}}{4}$

6 $\frac{dy}{dx} = x(3 + 4x)$

$y = \int 3x + 4x^2\ dx$

$y = \frac{3}{2}x^2 + \frac{4}{3}x^3 + c$

$x = 1,\ y = 2 \Rightarrow\ 2 = \frac{3}{2} + \frac{4}{3} + c$

$c = \frac{-5}{6}$

$\therefore\ y = \frac{3}{2}x^2 + \frac{4}{3}x^3 - \frac{5}{6}$

$x = -2,\ y = a\ \Rightarrow\ a = \frac{3}{2}(-2)^2 + \frac{4}{3}(-2)^3 - \frac{5}{6}$

$a = 6 - \frac{32}{3} - \frac{5}{6}$

$= -\frac{11}{2}$

Try these 15.3

(a) $\int \sin^5 x\ dx = \int \sin x\ (\sin^2 x)^2\ dx$

$= \int \sin x\ (1 - \cos^2 x)^2\ dx$

$= \int \sin x\ (1 - 2\cos^2 x + \cos^4 x)\ dx$

$= \int \sin x - 2\sin x\cos^2 x + \sin x\cos^4 x\ dx$

$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

(b) $\int \cos^5 x\ dx = \int \cos x\cos^4 x\ dx$

$$= \int \cos x\,(1 - \sin^2 x)^2\,dx$$

$$= \int \cos x\,(1 - 2\sin^2 x + \sin^4 x)\,dx$$

$$= \int \cos x - 2\cos x \sin^2 x + \cos x \sin^4 x\,dx$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

(c) $\displaystyle\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx, \quad \cos^2 x = \frac{1+\cos 2x}{2}$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\,dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right] + c$$

Try these 15.4

(a) $\displaystyle\int \tan^4 x\,dx = \int \tan^2 x \tan^2 x\,dx$

$$= \int \tan^2 x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x - \tan^2 x\,dx$$

$$= \int \tan^2 x \sec^2 x - (\sec^2 x - 1)]\,dx$$

$$= \frac{\tan^3 x}{3} - \tan x - x + c$$

(b) $\displaystyle\int \tan^5 x\,dx = \int \tan^3 x\,(\sec^2 x - 1)\,dx$

$$= \int (\tan^3 x \sec^2 x - \tan^3 x)\,dx$$

$$= \int \tan^3 x \sec^2 x - \tan x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^3 x \sec^2 x - \sec^2 x \tan x + \tan x\,dx$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + c$$

Try these 15.5

(a) $\displaystyle\int (\cos 6x \sin 3x)\,dx = \frac{1}{2}\int (\sin 9x - \sin 3x)\,dx$

$$= \frac{1}{2}\left[-\frac{1}{9}\cos 9x + \frac{1}{3}\cos 3x\right] + c$$

(b) $\displaystyle\int (\cos 8x \cos 2x)\,dx = \frac{1}{2}\int (\cos 10x + \cos 6x)\,dx$

$$= \frac{1}{2}\left[\frac{1}{10}\sin 10\,x + \frac{1}{6}\sin 6x\right] + c$$

(c) $\int (\sin 10x \sin x)\,dx = -\frac{1}{2}\int (\cos 11x - \cos 9x)\,dx$

$$= -\frac{1}{2}\left[\frac{1}{11}\sin(11x) - \frac{1}{9}\sin(9x)\right] + c$$

Exercise 15 C

1 $\int_1^4 f(x)\,dx = 8$

(a) $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -8$

(b) $\int_1^4 5f(x)\,dx = 5(8) = 40$

(c) $\int_1^3 f(x)\,dx + 4\int_1^3 x\,dx + \int_3^4 f(x)\,dx$

$= \int_1^4 f(x)\,dx + \left[\ 2x^2\right]_1^3$
$= 8 + (18 - 2)$
$= 8 + 16$
$= 24$

2 $\int_1^6 g(x)\,dx = 12$

(a) $2\int_1^6 g(x)\,dx = 2(12) = 24$

(b) $\int_1^6 3\,g(x) + 5\,dx$

$= 3\int_1^6 g(x)\,dx + \int_1^6 5\,dx$

$= 3[12] + \left[\ 5x\right]_1^6$
$= 36 + (30 - 5)$
$= 36 + 25$
$= 61$

(c) $\int_6^1 g(x)\,dx = -\int_1^6 g(x)\,dx = -12$

(d) $\int_1^6 [g(x) + Kx]\,dx = 47$

$\int_1^6 g(x)\,dx + K\int_1^6 x\,dx = 47$

$12 + \left[\frac{Kx^2}{2}\right]_1^6 = 47$

$18K - \frac{1}{2}K = 35$

$\frac{35K}{2} = 35$

$K = 2$

3 $\int_0^4 f(x)\,dx = 10, \quad \int_0^4 g(x)\,dx = 6$

(a) $\int_0^4 [f(x) + 3g(x)]\,dx$

$= \int_0^4 f(x)\,dx + 3\int_0^4 g(x)\,dx$

$= 10 + 3(6) = 28$

(b) $\int_0^4 f(x)\,f(x)\,dx$ cannot be evaluated

(c) $\int_0^4 f(x)\,dx + \int_4^0 g(x)\,dx = \int_0^4 f(x)dx - \int_0^4 g(x)dx$

$= 10 - 6$

$= 4$

(d) $\int_0^4 2g(x) + 3\,dx = 2\int_0^4 g(x)\,dx + \int_0^4 3\,dx$

$= 2(6) + [3x]_0^4$

$= 12 + 3(4) = 24$

(e) $\int_0^5 g(x)\,dx$ cannot be evaluated

4 $y = x(1 + x^2)^{½}$

$\frac{dy}{dx} = (1+x^2)^{1/2} + x\left(\frac{1}{2}\right)(2x)(1+x^2)^{-1/2}$

$= (1+x^2)^{1/2} + \frac{x^2}{\sqrt{1+x^2}}$

$= \frac{1+x^2+x^2}{\sqrt{1+x^2}}$

$= \frac{1+2x^2}{\sqrt{1+x^2}}$

Since $\frac{d}{dx}[x(1+x^2)^{\frac{1}{2}}] = \frac{1+2x^2}{\sqrt{1+x^2}}$

Integrating both sides wrt x from 0 to 1

$\Rightarrow \left[x(1+x^2)^{\frac{1}{2}}\right]_0^1 = \int_0^1 \frac{1+2x^2}{\sqrt{1+x^2}}\,dx$

$\Rightarrow 2^{1/2} = \int_0^1 \frac{1+2x^2}{\sqrt{1+x^2}}\,dx$

$\times 3 \Rightarrow 3\sqrt{2} = \int_0^1 \frac{3+6x^2}{(1+x^2)^{1/2}}\,dx$

5 $y = (1 + 4x)^{3/2}$

$\frac{dy}{dx} = \frac{3}{2}(1+4x)^{1/2}(4)$

$= 6(1 + 4x)^{1/2}$

Since $\frac{d}{dx}[(1+4x)^{3/2}] = 6(1+4x)^{1/2}$

Integrating both sides wrt x from 0 to 1

$\Rightarrow \left[(1+4x)^{\frac{3}{2}}\right]_0^1 = \int_0^1 6(1+4x)^{1/2}\,dx$

$$\Rightarrow [5^{\frac{3}{2}} - 1]\frac{1}{6} = \int_0^1 (1+4x)^{\frac{1}{2}}\,dx$$

$$\frac{1}{6}(5\sqrt{5} - 1) = \int_0^1 (1+4x)^{\frac{1}{2}}\,dx$$

6 $y = \dfrac{\sin x}{1+\cos x}$

$$\frac{dy}{dx} = \frac{(1+\cos x)\cos x - \sin x\,(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1+\cos x}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

$$\frac{d}{dx}\left[\frac{\sin x}{1+\cos x}\right] = \frac{1}{1+\cos x}$$

$$\Rightarrow \int_0^{\pi/4} \frac{1}{1+\cos x}\,dx = \left[\frac{\sin x}{1+\cos x}\right]_0^{\pi/4}$$

$$\Rightarrow \int_0^{\pi/4} \frac{1}{1+\cos x}\,dx = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$$

$$= \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{2\sqrt{2}-2}{2} = \sqrt{2} - 1$$

7 $y = \dfrac{x+1}{2-3x}$

$$\frac{dy}{dx} = \frac{(2-3x)(1) - (x+1)(-3)}{(2-3x)^2}$$

$$= \frac{2-3x+3x+3}{(2-3x)^2}$$

$$= \frac{5}{(2-3x)^2}$$

$$\frac{d}{dx}\left[\frac{x+1}{2-3x}\right] = \frac{5}{(2-3x)^2}$$

$$\Rightarrow \left[\frac{x+1}{2-3x}\right]_0^{1/2} = 5\int_0^{1/2} \frac{1}{(2-3x)^2}\,dx$$

$$\Rightarrow \left[\frac{3/2}{2-3/2}\right] - \left[\frac{1}{2}\right] = 5\int_0^{1/2} \frac{1}{(2-3x)^2}\,dx$$

$$\Rightarrow \frac{5}{2} \times \frac{1}{5} = \int_0^{1/2} \frac{1}{(2-3x)^2}\,dx = \frac{1}{2}$$

8 $2 \sin 5\theta \cos\theta = \sin (5\theta + \theta) + \sin (5\theta - \theta)$

$= \sin 6\theta + \sin 4\theta$

$$\int_0^{\pi/4} 2\sin 5\theta \cos\theta d\theta = \int_0^{\pi/4} (\sin 6\theta + \sin 4\theta)\,d\theta = \left[-\frac{1}{6}\cos 6\theta - \frac{1}{4}\cos 4\theta\right]_0^{\pi/4}$$

$$= \left(-\frac{1}{6}\cos\frac{6\pi}{4} - \frac{1}{4}\cos\frac{4\pi}{4}\right) - \left(-\frac{1}{6}\cos 0 - \frac{1}{4}\cos 0\right)$$

$$= \frac{1}{4} + \frac{1}{6} + \frac{1}{4} = \frac{2}{3}$$

9 $2 \sin 7\theta \cos 3\theta = \sin (7\theta + 3\theta) + \sin (7\theta - 3\theta)$

$= \sin (10\theta) + \sin (4\theta)$

$$\int_0^{\frac{\pi}{6}} 2\sin 7\theta \cos 3\theta\,d\theta = \int_0^{\pi/6} (\sin 10\theta + \sin 4\theta)\,d\theta$$

$$= \left[-\frac{1}{10}\cos 10\theta - \frac{1}{4}\cos 4\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left[-\frac{1}{10}\cos\frac{10\pi}{6} - \frac{1}{4}\cos\frac{4\pi}{6}\right] - \left[-\frac{1}{10}\cos 0 - \frac{1}{4}\cos 0\right]$$

$$= \frac{-1}{20} + \frac{1}{8} + \frac{1}{10} + \frac{1}{4} = \frac{17}{40}$$

10 $\displaystyle\int_{\frac{\pi}{4}}^{\pi/2} (\cos 6\theta \cos 2\theta)\,d\theta$

$2 \cos 6\theta \cos 2\theta = \cos (6\theta + 2\theta) + \cos (6\theta - 2\theta)$

$= \cos 8\theta + \cos 4\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 6\theta \cos 2\theta\,d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 8\theta + \cos 4\theta\,d\theta = \frac{1}{2}\left[\frac{1}{8}\sin 8\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{8}\sin 4\pi + \frac{1}{4}\sin 2\pi\right) - \left(\frac{1}{8}\sin 2\pi + \frac{1}{4}\sin \pi\right)\right]$$

$= 0$

11 $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin 5\theta \sin\theta\,d\theta$

$-2 \sin 5\theta \sin\theta = \cos (5\theta + \theta) - \cos (5\theta - \theta)$

$= \cos 6\theta - \cos 4\theta$

$2 \sin 5\theta \sin\theta = \cos 4\theta - \cos 6\theta$

$$\therefore \int_0^{\frac{\pi}{2}} 2\sin 5\theta \sin\theta\,d\theta = \int_0^{\frac{\pi}{2}} \cos 4\theta - \cos 6\theta)\,d\theta = \left[\frac{1}{4}\sin 4\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/2}$$

$$= \left(\frac{1}{4}\sin 2\pi - \frac{1}{6}\sin 3\pi\right) - \left(\frac{1}{4}\sin 0 - \frac{1}{6}\sin 0\right)$$

$= 0$

12 $-2 \sin 5\theta \sin 3\theta = \cos (5\theta + 3\theta) - \cos (5\theta - 3\theta)$

$= \cos 8\theta - \cos 2\theta$

$\therefore 2 \sin 5\theta \sin 3\theta = -\cos 8\theta + \cos 2\theta$

$$\int_{\pi/3}^{\pi/2} 2\sin 5\theta \sin 3\theta\,d\theta = \int_{\pi/3}^{\pi/2} -\cos 8\theta + \cos 2\theta\,d\theta = \left[-\frac{1}{8}\sin 8\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= \left(-\frac{1}{8}\sin 4\pi + \frac{1}{2}\sin\pi\right) - \left(-\frac{1}{8}\sin\frac{8\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{4} = \frac{-3\sqrt{3}}{16}$$

13 $2\cos 7\theta\cos 3\theta = \cos(7\theta + 3\theta) + \cos(7\theta - 3\theta) = \cos 10\theta + \cos 4\theta$

$$\cos 7\theta\cos 3\theta = \frac{1}{2}\cos 10\theta + \frac{1}{2}\cos 4\theta$$

$$\int_0^{\pi/12}(\cos 7\theta\cos 3\theta)\,d\theta = \int_0^{\pi/12}\left(\frac{1}{2}\cos 10\theta + \frac{1}{2}\cos 4\theta\right)d\theta = \left[\frac{1}{20}\sin 10\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/12}$$

$$= \frac{1}{20}\sin\frac{10\pi}{12} + \frac{1}{8}\sin\frac{4\pi}{12} = \frac{1}{40} + \frac{\sqrt{3}}{16}$$

14 $\int_0^{\pi/2}\sin x\,dx = [-\cos x]_0^{\pi/2}$

$$= -\cos\frac{\pi}{2} + \cos 0 = 1$$

15 $\int_0^{3\pi/2}\sin 3x\,dx = \left[-\frac{1}{3}\cos 3x\right]_0^{3\pi/2}$

$$= -\frac{1}{3}\cos\frac{9\pi}{2} + \frac{1}{3} = \frac{1}{3}$$

16 $\int_0^{\pi/4}(1 + \cos x)\,dx. = [x + \sin x]_0^{\pi/4}$

$$= \frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\pi}{4} + \frac{\sqrt{2}}{2}$$

17 $\int_0^{\pi/4}\cos 2x\,dx = \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$$= \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$$

18 $\int_0^{\pi/2}\cos 4x\,dx = \left[\frac{1}{4}\sin 4x\right]_0^{\pi/2}$

$$= \frac{1}{4}\sin 2\pi = 0$$

19 $\int_0^{\pi/2}\cos\left(7x + \frac{\pi}{2}\right)dx = \left[\frac{1}{7}\sin\left(7x + \frac{\pi}{2}\right)\right]_0^{\pi/2}$

$$= \frac{1}{7}\sin\frac{8\pi}{2} - \frac{1}{7}\sin\frac{\pi}{2}$$

$$= -\frac{1}{7}$$

20 $\int_0^{\pi/2}\sin^2 2x\,dx = \frac{1}{2}\int_0^{\pi/2} 1 - \cos 4x\,dx$ using $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right]$$

$$= \frac{\pi}{4}$$

21 $\int_0^{\pi/4} \cos^2 4x\, dx = \frac{1}{2}\int_0^{\pi/4} 1 + \cos 8x\, dx$ using $\cos^2 x = \frac{1+\cos 2x}{2}$

$$= \frac{1}{2}\left[x + \frac{1}{8}\sin 8x\right]_0^{\pi/4}$$ using $\cos^2 4x = \frac{1+\cos 8x}{2}$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{8}\sin 2\pi\right]$$

$$= \frac{\pi}{8}$$

22 $\int_0^{\pi/2} \cos^2 6x\, dx$

$$= \frac{1}{2}\int_0^{\pi/2} 1 + \cos 12x\, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{12}\sin 12x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{12}\sin 6\pi\right]$$

$$= \frac{\pi}{4}$$

23 $\int_0^{\pi/2} \sin^2\left(\frac{x}{2}\right) dx$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - \cos x\, dx$$

$$= \frac{1}{2}[x - \sin x]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \sin\frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right]$$

24 $\int_{\frac{\pi}{6}}^{\pi/3} \tan^2 (2x)\, dx$

$$\int_{\frac{\pi}{6}}^{\pi/3} [\sec^2 (2x) - 1]\, dx$$

$$= \left[\frac{1}{2}\tan(2x) - x\right]_{\frac{\pi}{6}}^{\pi/3}$$

$$= \left[\frac{1}{2}\tan\left(\frac{2\pi}{3}\right) - \frac{\pi}{3}\right] - \left[\frac{1}{2}\tan\left(\frac{2\pi}{6}\right) - \frac{\pi}{6}\right]$$

$$= \frac{-\sqrt{3}}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{6}$$

$$= -\sqrt{3} - \frac{\pi}{6}$$

25 (a) $\int_1^4 4x - 6\sqrt{x}\,dx$

$= \int_1^4 4x - 6x^{1/2}\,dx$

$= \left[\,2x^2 - 4x^{3/2}\right]_1^4$

$= [32 - 4(\sqrt{4})^3] - [2 - 4]$

$= 2$

(b) $\int_1^4 2\sqrt{x} - \frac{4}{\sqrt{x}}\,dx$

$= \int_1^4 2x^{1/2} - 4x^{-1/2}\,dx$

$= \left[\frac{4}{3}x^{3/2} - 8x^{1/2}\right]_1^4$

$= \left[\frac{4}{3}(8) - 16\right] - \left[\frac{4}{3} - 8\right]$

$= \frac{32}{3} - \frac{48}{3} - \frac{4}{3} + \frac{24}{3}$

$= \frac{4}{3}$

26 (a) $\int_{-1}^0 \frac{4}{\sqrt{1-3x}}\,dx$

$= \int_{-1}^0 4(1-3x)^{-1/2}\,dx$

$= \left[\frac{4}{-3} \times 2(1-3x)^{1/2}\right]_{-1}^0$

$= \frac{-8}{3}[1-2]$

$= \frac{8}{3}$

(b) $\int_0^2 \sqrt{1+4x}\,dx = \int_0^2 (1+4x)^{1/2}\,dx$

$= \left[\frac{1}{4} \times \frac{2}{3}(1+4x)^{3/2}\right]_0^2$

$= \frac{1}{6}[9^{3/2} - 1] = \frac{1}{6}(26) = \frac{13}{3}$

27 $\cos 3x = \cos(2x + x)$

$= \cos 2x \cos x - \sin 2x \sin x$

$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$

$= 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x)$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$= 4\cos^3 x - 3\cos x$

$4\cos^3 x = \cos 3x + 3\cos x$

$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

$\int_{\pi/4}^{\pi/2} \cos^3 x\,dx = \int_{\pi/4}^{\pi/2} \frac{1}{4}\cos 3x + \frac{3}{4}\cos x\,dx$

$$=\left[\frac{1}{12}\sin 3x+\frac{3}{4}\sin x\right]_{\pi/4}^{\pi/2}$$

$$=\left[\frac{1}{12}\sin\frac{3\pi}{2}+\frac{3}{4}\sin\frac{\pi}{2}\right]-\left[\frac{1}{12}\sin\frac{3\pi}{4}+\frac{3}{4}\sin\frac{\pi}{4}\right]$$

$$=\frac{-1}{12}+\frac{3}{4}-\frac{\sqrt{2}}{24}-\frac{3\sqrt{2}}{8}=\frac{2}{3}+\frac{\sqrt{2}}{3}=\frac{1}{3}(2+\sqrt{2})$$

28 (a) $\int_1^5 (3x+1)^{-1/2}\,dx$

$$=\left[\frac{2}{3}(3x+1)^{1/2}\right]_1^5$$

$$=\frac{2}{3}[16^{1/2}-4^{1/2}]$$

$$=\frac{2}{3}[4-2]=\frac{4}{3}$$

(b) $\int_0^{\pi/2} (\sin 3x-\cos 2x)\,dx$

$$=\left[-\frac{1}{3}\cos 3x-\frac{1}{2}\sin 2x\right]_0^{\pi/2}$$

$$=\left(-\frac{1}{3}\cos\frac{3\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(-\frac{1}{3}\right)$$

$$=\frac{1}{3}$$

29 (a) $\int_1^9 (1+7x)^{-1/3}\,dx$

$$=\left[\frac{1}{7}\frac{(1+7x)^{2/3}}{2/3}\right]_1^9$$

$$=\frac{3}{14}[64^{2/3}-8^{2/3}]$$

$$=\frac{3}{14}[16-4]$$

$$=\frac{18}{7}$$

(b) $\int_4^{15} (x-3)^{-3/2}\,dx$

$$=\left[-2(x-3)^{-1/2}\right]_4^{15}$$

$$=-2\left[\frac{1}{\sqrt{12}}-1\right]$$

$$=2-\frac{\sqrt{12}}{6}=2-\frac{\sqrt{3}}{3}$$

30 $\sin 3x = \sin(2x + x)$
$= \sin 2x \cos x + \cos 2x \sin x$
$= 2\sin x\cos^2 x + (1 - 2\sin^2 x)\sin x$
$= 2\sin x\,[1 - \sin^2 x] + \sin x - 2\sin^3 x$
$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
$= 3\sin x - 4\sin^3 x$

$4\sin^3 x = 3\sin x - \sin 3x$

$\int_0^{\pi/4} 4\sin^3 x\,dx = \int_0^{\pi/4} 3\sin x - \sin 3x\,dx$

$= \left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\pi/4}$

$= \left(-3\cos\frac{\pi}{4} + \frac{1}{3}\cos\frac{3\pi}{4}\right) - \left(-3 + \frac{1}{3}\right)$

$= \frac{-3\sqrt{2}}{2} - \frac{\sqrt{2}}{6} + 3 - \frac{1}{3} = \frac{-5}{3}\sqrt{2} + \frac{8}{3} = \frac{1}{3}(8 - 5\sqrt{2})$

Exercise 15 D

1 $\int x^2 (x^3 + 1)\,dx$

$u = x^3 + 1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$\int x^2(x^3+1)^3\,dx = \int \frac{1}{3}u\,du$

$= \frac{1}{6}u^2 + c$

$= \frac{1}{6}(x^3+1)^2 + c$

2 $\int \frac{x+1}{(6x^2+12x+5)^4}\,dx$

$u = 6x^2 + 12x + 5$

$\frac{du}{dx} = 12x + 12$

$\frac{1}{12}du = (x+1)\,dx$

$(6x^2 + 12x + 5)^4 = u^4$

$\int \frac{x+1}{(6x^2+12x+5)^4}\,dx = \frac{1}{12}\int \frac{1}{u^4}\,du$

$= \frac{1}{12}\int u^{-4}\,du$

$= \frac{1}{12}\left[\frac{u^{-3}}{-3}\right] + c$

$= -\frac{1}{12u^3} + c$

$= -\frac{1}{12(6x^2+12x+5)^3} + c$

3 $\int \cos^4 x \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$\cos^4 x = u^4$

$\int \cos^4 x \sin x\, dx = -\int u^4\, du$

$= -\frac{u^5}{5} + c$

$= -\frac{\cos^5 x}{5} + c$

4 $\int \cos 4x \sin^3 4x\, dx$

$u = \sin 4x$

$du = 4 \cos 4x\, dx$

$\frac{1}{4} du = \cos 4x\, dx$

$\int \cos 4x \sin^3 4x\, dx = \frac{1}{4}\int u^3\, du$

$= \frac{1}{16} u^4 + c$

$= \frac{1}{16} \sin^4 (4x) + c$

5 $\int \frac{6x}{\sqrt{3x+1}}\, dx$

$u = 3x + 1$

$du = 3\, dx$

$\frac{1}{3} du = dx$

Since $3x = u - 1$

$6x = 2u - 2$

$\int \frac{6x}{\sqrt{3x+1}}\, dx$

$= \frac{1}{3}\int \frac{2u-2}{\sqrt{u}}\, du$

$= \frac{2}{3}\int u^{1/2} - u^{-1/2}\, du$

$= \frac{2}{3}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right] + c$

$= \frac{4}{9}(3x+1)^{3/2} - \frac{4}{3}(3x+1)^{1/2} + c$

6 $\int_0^1 \frac{x}{(7x+2)^3}\, dx$

$u = 7x + 2$

$du = 7dx$

$\frac{1}{7} du = dx$

$7x = u - 2$

$x = \frac{u-2}{7}$

$(7x + 2)^3 = u^3$

When $x = 0$, $u = 2$

x = 1, u = 9

$$\int_0^1 \frac{x}{(7x+2)^3}\,dx = \frac{1}{7}\int_2^9 \frac{\frac{u-2}{7}}{u^3}\,du$$

$$= \frac{1}{49}\int_2^9 u^{-2} - 2u^{-3})\,du$$

$$= \frac{1}{49}\left[-\frac{1}{u} + \frac{1}{u^2}\right]_2^9$$

$$= \frac{1}{49}\left[\left(-\frac{1}{9} + \frac{1}{81}\right) - \left(-\frac{1}{2} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{49}\left[\frac{-8}{81} + \frac{1}{4}\right]$$

$$= \frac{1}{324}$$

7 $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$

u = tan x

du = $\sec^2 x$ dx

$\tan^3 x = u^3$

$x = \frac{\pi}{4},\ u = \tan\frac{\pi}{4} = 1$

x = 0, u = tan 0 = 0

$$\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx = \int_0^1 u^3 du = \left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{4}$$

8 $\int_0^{\pi/4} 2\sin^4 x \cos x\,dx$

u = sin x

du = cos x dx

$\sin^4 x = u^4$

$x = \frac{\pi}{4},\ u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

x = 0, u = sin 0 = 0

$$\int_0^{\pi/4} 2\sin^4 x \cos x\,dx = 2\int_0^{\sqrt{2}/2} u^4\,du$$

$$= \left[\frac{2}{5}u^5\right]_0^{\sqrt{2}/2}$$

$$= \frac{2}{5}\left(\frac{\sqrt{2}}{2}\right)^5 = \frac{4\sqrt{2}\times 2}{5\times 32} = \frac{\sqrt{2}}{20}$$

9 $\int_0^1 \frac{x+2}{\sqrt{x+1}}\,dx$

u = x + 1

du = dx

u + 1 = x + 2

$\sqrt{x+1} = \sqrt{u} = u^{1/2}$

x = 1, u = 2

x = 0, u = 1

$$\int_0^1 \frac{x+2}{\sqrt{x+1}}\,dx$$

$$= \int_1^2 \frac{u+1}{\sqrt{u}}\,du$$

$$= \int_1^2 u^{1/2} + u^{-1/2}\,du$$

$$= \left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^2$$

$$= \left(\frac{4}{3}\sqrt{2} + 2\sqrt{2}\right) - \left(\frac{2}{3} + 2\right)$$

$$= \frac{10\sqrt{2}}{3} - \frac{8}{3}$$

10 $\int_{\pi/4}^{\pi/3} \sin^2 x \cos x\,dx$

$u = \sin x$
$du = \cos x\,dx$
$\sin^2 x = u^2$

$$x = \frac{\pi}{3},\ u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{4},\ u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\int_{\pi/4}^{\pi/3} \sin^2 x \cos x\,dx = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} u^2\,du = \left[\frac{1}{3}u^3\right]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3$$

$$= \frac{1}{3}\,\frac{3\sqrt{3}}{8} - \frac{\sqrt{2}}{12} = \frac{\sqrt{3}}{8} - \frac{\sqrt{2}}{12}$$

$$= \frac{3\sqrt{3} - 2\sqrt{2}}{24}$$

$$= \frac{(3\sqrt{3} - 2\sqrt{2})\,(3\sqrt{3} + 2\sqrt{2})}{24(3\sqrt{3} + 2\sqrt{2})}$$

$$= \frac{27 - 8}{24\,(3\sqrt{3} + 2\sqrt{2})} = \frac{19}{24\,(3\sqrt{3} + 2\sqrt{2})}$$

11 (a) $\int x \cos x^2\,du$

$u = x^2$
$du = 2x\,dx$

$$\frac{1}{2}du = x\,dx$$

$\cos x^2 = \cos u$

$$\int x \cos x^2\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + c$$

$$= \frac{1}{2}\sin x^2 + c$$

(b) $\int x^2 \sin(x^3)\,dx$

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$\int x^2 \sin x^3\,dx = \frac{1}{3}\int \sin u\,du$

$= -\frac{1}{3}\cos u + c$

$= -\frac{1}{3}\cos x^3 + c$

(c) $\int \frac{2-3x}{(4+x)^4}\,dx$

$u = x + 4$

$du = dx$

$x = u - 4$

$2 - 3x = 2 - 3(u - 4) = 14 - 3u$

$\int \frac{2-3x}{(4+x)^4}\,dx = \int \frac{14-3u}{u^4}\,du$

$= \int 14u^{-4} - 3u^{-3}\,du$

$= \frac{14}{-3u^3} + \frac{3}{2u^2} + c$

$= -\frac{14}{3(x+4)^3} + \frac{3}{2(x+4)^2} + c$

Exercise 15 E

1 $\frac{dy}{dx} = 6x^2 - 4$

$y = \int 6x^2 - 4\,dx$

$y = 2x^3 - 4x + c$

$x = 1, y = 2 \Rightarrow 2 = 2 - 4 + c$

$c = 4$

$y = 2x^3 - 4x + 4$

2 (a) $\frac{dy}{dx} = px - 5$

$px - 5 = 4$

$x = -3 \Longrightarrow -3p = 9$

$p = -3$

$\frac{dy}{dx} = -3x - 5$

(b) $y = \int -3x - 5\,dx$

$y = \frac{-3}{2}x^2 - 5x + c$

$x = -3,\ y = 2 \Rightarrow\ 2 = \frac{-27}{2} + 15 + c$

$c = 2 - \frac{3}{2} = \frac{1}{2}$

$y = \frac{-3}{2}x^2 - 5x + \frac{1}{2}$

3 $\frac{dy}{dx} = \frac{6}{\sqrt{4x+1}}$

$y = \int 6\,(4x+1)^{-1/2}\,dx$

$y = \frac{6}{4}\frac{(4x+1)^{1/2}}{1/2} + c$

$y = 3\,(4x + 1)^{1/2} + c$

$x = 2,\ y = 6 \Rightarrow 6 = 3(9)^{1/2} + c$

$c = -3$

$y = 3\,(4x + 1)^{1/2} - 3$

$x = 6,\ y = h \Rightarrow h = 3\,(25)^{1/2} - 3$

$= 15 - 3 = 12$

4 $\frac{dy}{dx} = 3x^2 + \frac{2}{x^2}$

$y = \int 3x^2 + 2x^{-2}\,dx$

$y = x^3 - \frac{2}{x} + c$

$x = 1,\ y = 4 \Rightarrow 4 = 1 - 2 + c$

$c = 5$

$y = x^3 - \frac{2}{x} + 5$

5 $\frac{dy}{dx} = \sqrt{1+8x}$

$y = \int (1+8x)^{1/2}\,dx$

$y = \frac{1}{8}\frac{(1+8x)^{3/2}}{3/2} + c$

$y = \frac{1}{12}(1+8x)^{3/2} + c$

$x = 3,\ y = 8 \Rightarrow\ 8 = \frac{1}{12}(25)^{3/2} + c$

$8 = \frac{125}{12} + c$

$c = -\frac{29}{12}$

$y = \frac{1}{12}(1+8x)^{3/2} - \frac{29}{12}$

6 $\frac{dy}{dx} = (4x-6)^3$

$y = \int (4x-6)^3\,dx$

$$y = \frac{1}{4}\frac{(4x-6)^4}{4} + c$$

$$y = \frac{1}{16}(4x-6)^4 + c$$

$$x = 1, y = 2 \Rightarrow 2 = \frac{1}{16}(16) + c$$

$c = 1$

$$y = \frac{1}{16}(4x-6)^4 + 1$$

7 $$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

$$y = \int 1 - 4x^{-2}\, dx$$

$$y = x + \frac{4}{x} + c$$

$x = 4, y = 6 \Rightarrow 6 = 4 + 1 + c$
$c = 1$

$$y = x + \frac{4}{x} + 1$$

8 $$\frac{dy}{dx} = 4x^3 - 2x + 1$$

$$y = \int 4x^3 - 2x + 1\, dx$$

$y = x^4 - x^2 + x + c$
$x = 4, y = 5 \Rightarrow 5 = 4^4 - 4^2 + 4 + c$
$c = -239$
$y = x^4 - x^2 + x - 239$

9 $f'(x) = 3x^3 + 6x^2 - 2x + k$

$$f(x) = \int 3x^3 + 6x^2 - 2x + k\, dx$$

$$f(x) = \frac{3}{4}x^4 + 2x^3 - x^2 + kx + c$$

$x = 0, f(x) = 4 \Rightarrow 4 = c$

$$x = 1, f(1) = -2 \Rightarrow -2 = \frac{3}{4} + 2 - 1 + k + 4$$

$$k = -2 - 5\frac{3}{4} = \frac{-31}{4}$$

$$f(x) = \frac{3}{4}x^4 + 2x^3 - x^2 - \frac{31}{4}x + 4$$

10 $$\frac{dy}{dx} = 2x^4 + 3x^3$$

$$y = \int 2x^4 + 3x^3\, dx$$

$$y = \frac{2}{5}x^5 + \frac{3}{4}x^4 + c$$

$$x = 2, y = -2 \Rightarrow -2 = \frac{64}{5} + 12 + c$$

$$c = \frac{-10}{5} - \frac{64}{5} - \frac{60}{5}$$

$= -\frac{134}{5}$

$y = \frac{2}{5}x^5 + \frac{3}{4}x^4 - \frac{134}{5}$

Review Exercise 15

1 $\int_1^9 f(x)\,dx = 42$

(a) $\int_1^9 6f(x)\,dx = (42)(6) = 252$

(b) $\int_1^9 (f(x) - 4)\,dx = \int_1^9 f(x)\,dx - \int_1^9 4\,dx$

$= 42 - [4x]_1^9$

$= 42 - [36 - 4] = 10$

2 $\int_0^3 g(x)\,dx = 12$

(a) $\int_0^3 5g(x)\,dx = 5(12) = 60$

(b) $\int_0^3 g(x) + 2]\,dx = \int_0^3 g(x)\,dx + \int_0^3 2\,dx$

$= 12 + [2x]_0^3 = 12 + 6 = 18$

(c) $\int_3^0 [g(x) + x]\,dx = -\left(\int_0^3 g(x)\,dx + \int_0^3 x\,dx\right)$

$= -12 - \left[\frac{1}{2}x^2\right]_0^3$

$= -12 - \frac{9}{2} = -\frac{33}{2}$

3 $\int_1^2 \frac{t^3 + 4t^2}{t^6}\,dt = \int_1^2 t^{-3} + 4t^{-4}\,dt$

$= \left[\frac{-1}{2t^2} - \frac{4}{3t^3}\right]_1^2$

$= \left(-\frac{1}{8} - \frac{1}{6}\right) - \left(-\frac{1}{2} - \frac{4}{3}\right)$

$= \frac{37}{24}$

4 $\int_0^1 (x + 1)(2x - 3)\,dx$

$= \int_0^1 2x^2 - x - 3\,dx$

$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x\right]_0^1$

$= \frac{2}{3} - \frac{1}{2} - 3$

$= \frac{-17}{6}$

5 (a) $\int (4x-1)^{1/2}\,dx$

$= \frac{1}{4}\frac{(4x-1)^{3/2}}{3/2} + c$

$= \frac{1}{6}(4x-1)^{3/2} + c$

(b) $\int 4(2-3t)^{-1/2}\,dt$

$= 4\left(\frac{1}{-3}\right)\frac{(2-3t)^{1/2}}{1/2} + c$

$= -\frac{8}{3}\sqrt{2-3t} + c$

6 $\int_{\pi/4}^{\pi/2} \cos^2(2x)\,dx = \frac{1}{2}\int_{\pi/4}^{\pi/2} 1+\cos(4x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_{\pi/4}^{\pi/2}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right) - \left(\frac{\pi}{4} + \frac{1}{4}\sin \pi\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8}$

7 (a) $\int \sin 6x \cos 4x\,dx = \frac{1}{2}\int \sin 10x + \sin 2x\,dx = -\frac{1}{20}\cos 10x - \frac{1}{4}\cos 2x + c$

using $2\sin 6x\cos 4x = \sin 10x + \sin 2x$

(b) $\int \sin 8x \sin 4x\,dx = \int \frac{-1}{2}\cos 12x + \frac{1}{2}\cos 4x\,dx = -\frac{1}{24}\sin 12x + \frac{1}{8}\sin 4x + c$

using $-2\sin 8x\sin 4x = \cos 12x - \cos 4x$

$\sin 8x \sin 4x = \frac{-1}{2}\cos 12x + \frac{1}{2}\cos 4x$

8 $y = \frac{x}{1+2x}$

$\frac{dy}{dx} = \frac{(1+2x)-(x)(2)}{(1+2x)^2}$

$= \frac{1}{(1+2x)^2}$

Since $\frac{d}{dx}\left[\frac{x}{1+2x}\right] = \frac{1}{(1+2x)^2}$

$\Rightarrow \frac{x}{1+2x} + c = \int \frac{1}{(1+2x)^2}\,dx$

9 $\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$

$= \frac{\sin x(-\sin x) - (\cos x)\cos x}{\sin^2 x}$

$= \frac{-[\sin^2 x + \cos^2 x]}{\sin^2 x}$

$$= \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}[\cot 2x] = -2\operatorname{cosec}^2(2x)$$

$\cot^2 2x = \operatorname{cosec}^2 2x - 1$

$$\int \cot^2 2x\, dx = \int \operatorname{cosec}^2 2x - 1\, dx = -\frac{1}{2}\cot 2x - x + c$$

10 $$\int_0^{\pi/2} (\sin x + \cos x)^2\, dx = \int_0^{\pi/2} \sin^2 x + 2\sin x \cos x + \cos^2 x\, dx$$

$$= \int_0^{\pi/2} 1 + \sin 2x\, dx$$

$$= \left[x - \frac{1}{2}\cos 2x\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} - \frac{1}{2}\cos \pi\right) - \left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2} = \frac{\pi}{2} + 1$$

11 $$y = \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{(1+x^2) - (x)(2x)}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

Since $\frac{d}{dx}\left[\frac{x}{1+x^2}\right] = \frac{1-x^2}{(1+x^2)^2}$

$$\Rightarrow \frac{4x}{1+x^2} + c = \int \frac{4-4x^2}{(1+x^2)^2}\, dx$$

12 $$\frac{d}{dx}[x \sin x] = x\cos x + \sin x$$

Integrate both sides from 0 to $\frac{\pi}{4}$

$$\Rightarrow [x \sin x]_0^{\pi/4} = \int_0^{\pi/4} x\cos x + \sin x\, dx$$

$$\Rightarrow \frac{\pi}{4}\sin\frac{\pi}{4} = \int_0^{\pi/4} x\cos x\, dx + \int_0^{\pi/4} \sin x\, dx$$

$$\Rightarrow \frac{\sqrt{2}\pi}{8} = \int_0^{\pi/4} x\cos x\, dx + [-\cos x]_0^{\pi/4}$$

$$\frac{\sqrt{2}}{8}\pi = \int_0^{\pi/4} x\cos x\, dx + \left[-\cos\frac{\pi}{4} + \cos 0\right]$$

$$\int_0^{\pi/4} x \cos x\, dx = \frac{\sqrt{2}}{8}\pi + \frac{\sqrt{2}}{2} - 1$$

13 (a) $$y = \frac{x}{x^2 + 32}$$

$$\frac{dy}{dx} = \frac{(x^2 + 32) - x(2x)}{(x^2 + 32)^2}$$

$$= \frac{32 - x^2}{(x^2 + 32)^2}$$

(b) $$\int_0^1 \frac{32 - x^2}{(x^2 + 32)^2}\,dx = \left[\frac{x}{x^2 + 32}\right]_0^1$$

$$= \frac{1}{33}$$

14 $$f(x) = \frac{2x}{1 - 3x}$$

$$f'(x) = \frac{(1 - 3x)(2) - 2x(-3)}{(1 - 3x)^2}$$

$$= \frac{1}{(1 - 3x)^2}$$

$$\frac{d}{dx}\left[\frac{2x}{1 - 3x}\right] = \frac{1}{(1 - 3x)^2}$$

$$\Rightarrow \left[\frac{2x}{1 - 3x}\right]_0^1 = \int_0^1 \frac{1}{(1 - 3x)^2}\,dx$$

$$\Rightarrow -1 = \int_0^1 \frac{1}{(1 - 3x)^2}\,dx$$

$$\Rightarrow -6 = \int_0^1 \frac{6}{(1 - 3x)^2}\,dx$$

15 $$y = \frac{x}{(2x^2 + 3)^{1/2}}$$

$$\frac{dy}{dx} = \frac{(2x^2 + 3)^{1/2} - x\left(\frac{1}{2}\right)(4x)(2x^2 + 3)^{-1/2}}{2x^2 + 3}$$

$$= \frac{\sqrt{2x^2 + 3} - \dfrac{2x^2}{\sqrt{2x^2 + 3}}}{2x^2 + 3}$$

$$= \frac{3}{(2x^2 + 3)^{3/2}}$$

$$\frac{2x}{(2x^2 + 3)^{1/2}} + c = \int \frac{6}{(2x^2 + 3)^{3/2}}\,dx$$

16 $$y = \frac{5x}{3x^2 + 1}$$

$$\frac{dy}{dx} = \frac{(3x^2 + 1)(5) - 5x(6x)}{(3x^2 + 1)^2}$$

$$= \frac{5 - 15x^2}{(3x^2 + 1)^2}$$

$$\frac{d}{dx}\left[\frac{5x}{3x^2+1}\right]=\frac{5(1-3x^2)}{(3x^2+1)^2}$$

$$\Rightarrow\left[\frac{x}{3x^2+1}\right]_0^1=\int_0^1\frac{1-3x^2}{(1+3x^2)^2}\,dx$$

$$\Rightarrow\frac{1}{4}=\int_0^1\frac{1-3x^2}{(1+3x^2)^2}\,dx$$

17 (a) $\frac{dy}{dx}=4x^3-6x+2$

$y=\int 4x^3-6x+2\,dx$

$y = x^4 - 3x^2 + 2x + c$

$x = 1, y = -5 \Rightarrow -5 = 1 - 3 + 2 + c$

$c = -5$

$y = x^4 - 3x^2 + 2x - 5$

(b) when $x = 2$, $y = 2^4 - 3(2)^2 + 2(2) - 5$

$= 16 - 12 + 4 - 5$

$= 3$

$\frac{dy}{dx}=4(2)^3-6(2)+2=22$

Equation of the tangent at (2, 3) is

$y - 3 = 22(x - 2)$

$y = 22x - 41$

18 $\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{x}{\sqrt{1-x^2}}\,dx$

$u = 1 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$x=\frac{3}{4},\ u=1-\frac{9}{16}=\frac{7}{16}$

$x=\frac{1}{2},\ u=1-\frac{1}{4}=\frac{3}{4}$

$-\frac{1}{2}\int_{3/4}^{7/16}u^{-1/2}\,du$

$=\left[-\sqrt{u}\right]_{3/4}^{7/16}$

$=-\sqrt{\frac{7}{16}}+\sqrt{\frac{3}{4}}$

$= 0.205$

19 $\int_0^1\frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\sqrt{4-x^2}=\sqrt{4-4\sin^2\theta}$

$=\sqrt{4(1-\sin^2\theta)}$

$= \sqrt{4\cos^2\theta} = 2\cos\theta$

$x^2 = 4\sin^2\theta$

when $x = 0 \Rightarrow \sin\theta = 0, \theta = 0$

$x = 1, \sin\theta = \frac{1}{2}, \theta = \frac{\pi}{6}$

$\therefore \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

$\int_0^{\pi/6} \frac{4\sin^2\theta}{\cancel{2\cos\theta}} \cancel{(2\cos\theta)}\,d\theta$

$\int_0^{\pi/4} 4\sin^2\theta\,d\theta$

$= \int_0^{\pi/4} 4\left[\frac{1-\cos 2\theta}{2}\right]d\theta$

$= 2\int_0^{\pi/4} 1-\cos 2\theta\,d\theta$

$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{\pi}{2} - 1$

20 $\frac{dy}{dx} = \frac{8}{(4x-5)^2}$

$y = \int 8(4x-5)^{-2}\,dx$

$y = -\frac{2}{4x-5} + c$

$x = 2, y = 6 \Rightarrow 6 = \frac{-2}{3} + c$

$c = \frac{20}{3}$

$y = \frac{20}{3} - \frac{2}{4x-5}$

$y = 0 \Rightarrow \frac{2}{4x-5} = \frac{20}{3}$

$\frac{3}{10} = 4x - 5$

$4x = 5\frac{3}{10}$

$x = \frac{53}{40}$

$\left(\frac{53}{40}, 0\right)$

21 $\int \frac{\sin^3 x}{(1+\cos x)^4}\,dx$

u = 1 + cos x

du = – sin x dx

$\sin^2 x = 1 - \cos^2 x$

$= 1 - (u-1)^2$

$= 1 - u^2 + 2u - 1$

$= 2u - u^2$

$$\int \frac{\sin^3 x}{(1+\cos x)^4}\,dx = -\int \frac{2u-u^2}{u^4}\,du$$

$$= \int \frac{u^2-2u}{u^4}\,du$$

$$= \int \frac{u-2}{u^3}\,du$$

$$= \int \frac{1}{u^2} - \frac{2}{u^3}\,du$$

$$= \int u^{-2} - 2u^{-3}\,du = -\frac{1}{u} + \frac{1}{u^2} + c$$

Since u = 1 + cos x $\Rightarrow \int \frac{\sin^3 x}{(1+\cos x)^4}\,dx = -\frac{1}{(1+\cos x)} + \frac{1}{(1+\cos x)^2} + c$

22 $\int_0^1 \frac{x-1}{(3x^2-6x+5)^4}\,dx$

$u = 3x^2 - 6x + 5$

du = (6x – 6) dx

x = 1, u = 3 – 6 + 5 = 2

x = 0, u = 5

$$\int_0^1 \frac{x-1}{(3x^2-6x+5)^4}\,dx = \frac{1}{6}\int_5^2 \frac{1}{u^4}\,du$$

$$= -\frac{1}{6}\int_2^5 \frac{1}{u^4}\,du$$

$$= \frac{1}{18}\left[\frac{1}{u^3}\right]_2^5$$

$$= \frac{1}{18}\left[\frac{1}{125} - \frac{1}{8}\right]$$

= – 0.0065

23 $\int_0^{\pi/6} \frac{\sin\theta}{\cos^3\theta}\,d\theta$

u = cosθ, du = – sinθ dθ

$\theta = \pi/6, \quad u = \cos\pi/6 = \frac{\sqrt{3}}{2}$

$\theta = 0, \quad u = \cos 0 = 1$

$$\int_0^{\pi/6} \frac{\sin\theta}{\cos^3\theta}\,d\theta = -\int_1^{\frac{\sqrt{3}}{2}} \frac{1}{u^3}\,du = \left[\frac{1}{2u^2}\right]_1^{\frac{\sqrt{3}}{2}} = \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6}$$

24 $\int_0^{1/2} \frac{x}{(1-x^2)^2}\, dx$

$u = 1 - x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

$x = \frac{1}{2},\ u = 1 - \frac{1}{4} = \frac{3}{4}$

$x = 0,\ u = 1$

$-\frac{1}{2}\int_1^{3/4} \frac{1}{u^2}\, du$

$= \left[\frac{1}{2u}\right]_1^{3/4} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

25 $\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}}\, dx$

$x = \tan u \quad \Rightarrow dx = \sec^2 u\, du$

$(1+x^2)^{3/2} = (1+\tan^2 u)^{3/2}$

$= (\sec^2 u)^{3/2}$

$= \sec^3 u$

When $x = \sqrt{3},\ u = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$x = 0,\ u = \tan^{-1}(0) = 0$

$\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{3/2}}\, dx = \int_0^{\pi/3} \frac{\sec^2 u}{\sec^3 u}\, du$

$= \int_0^{\pi/3} \frac{1}{\sec u}\, du$

$= \int_0^{\pi/3} \cos u\, du$

$= [\sin u]_0^{\pi/3}$

$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$