# Chapter 14 Applications of Differentiation

## Try these 14.1

(a) $y=\dfrac{x-1}{x+1}$

$u = x - 1,\ v = x + 1$

$\dfrac{du}{dx}=1, \dfrac{dv}{dx}=1$

$\dfrac{dy}{dx}=\dfrac{(x+1)-(x-1)}{(x+1)^2}$

$=\dfrac{2}{(x+1)^2}$

$x=2,\ y=\dfrac{2-1}{2+1}=\dfrac{1}{3},\ \dfrac{dy}{dx}=\dfrac{2}{(2+1)^2}=\dfrac{2}{9}$

Equation of the tangent is:

$y-\dfrac{1}{3}=\dfrac{2}{9}(x-2)$

$y=\dfrac{2}{9}x-\dfrac{4}{9}+\dfrac{1}{3}$

$y=\dfrac{2}{9}x-\dfrac{1}{9}$

The equation of the tangent at $x = 2$ is

$y=\dfrac{2}{9}x-\dfrac{1}{9}$

(b) $y = x^2 \sin x$

$u = x^2,\ v = \sin x$

$\dfrac{du}{dx}=2x, \dfrac{dv}{dx}=\cos x$

$\dfrac{dy}{dx}=x^2\cos x+2x\sin x$

when $x = 0,\ y = 0,\ \dfrac{dy}{dx}=0$

Gradient of the normal $\to\infty$

$\therefore$ Equation of the normal at $x = 0$ is

$x = 0$

## Try these 14.2

(a) (i) $f(x) = x^2 + 2x + 3$

$f'(x)=2x+2$

$f'(x)>0$

$2x + 2 > 0$

$x > -1$

$\therefore$ increasing for $\{x : x \geq -1\}$

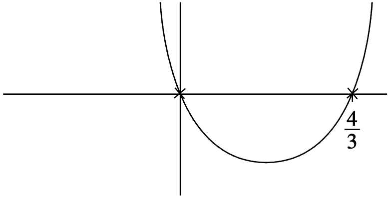
(ii) $f(x) = x^3 - 2x^2 + 5$

$f'(x) = 3x^2 - 4x$

$f'(x) > 0$

$3x^2 - 4x > 0$

$x(3x - 4) > 0$



$\{x : x \le 0\} \cup \left\{x : x \ge \frac{4}{3}\right\}$

(iii) $f(x) = x^4 - x$

$f'(x) = 4x^3 - 1$

$f'(x) > 0 \Rightarrow 4x^3 - 1 > 0$

$x > \sqrt[3]{\frac{1}{4}} = 0.63$

$\therefore \{x : x \ge 0.63\}$

(b) (i) $f(x) = 4x^2 + 6x + 2$

$f'(x) = 8x + 6$

$f'(x) < 0$

$8x + 6 < 0$

$x < \frac{-3}{4}$

$\left\{x: x \le \frac{-3}{4}\right\}$

(ii) $f(x) = \frac{x+1}{x-2}$

$f'(x) = \frac{(x-2)(1) - (x+1)(1)}{(x-2)^2}$

$= \frac{-3}{(x-2)^2}$

$f'(x) < 0 \quad \forall\, x \text{ since } (x-2)^2 > 0$

**Exercise 14A**

**1** $y = 6x^2 - 2x$

$\frac{dy}{dx} = 12x - 2$

$\frac{dy}{dx} > 0 \Rightarrow 12x - 2 > 0$

$x > \frac{1}{6}$

The function is increasing for $\left\{x : x \ge \frac{1}{6}\right\}$

**2** $x = t^4 - t^2$

$\frac{dx}{dt} = 4t^3 - 2t$

$= 2t(2t^2 - 1)$

$\frac{dx}{dt} < 0 \quad \Rightarrow 2t(2t^2 - 1) < 0$

$t(2t^2 - 1) < 0$

Critical values are 0, $\pm\frac{1}{\sqrt{2}}$

$t < -\frac{1}{\sqrt{2}}, \ t(2t^2 - 1) < 0$

$-\frac{1}{\sqrt{2}} < t < 0, t(2t^2 - 1) > 0$

$0 < t < \frac{1}{\sqrt{2}}, \ t(2t^2 - 1) < 0$

$\frac{1}{\sqrt{2}} < t, t(2t^2 - 1) > 0$

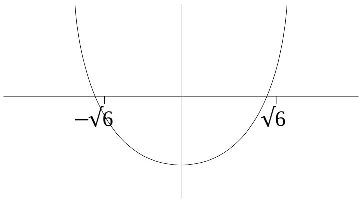
The function is decreasing for $\left\{t : t \le -\frac{1}{\sqrt{2}}\right\} \cup \left\{t : 0 \le t \le \frac{1}{\sqrt{2}}\right\}$

**3**

$y = \frac{1}{3}x + \frac{2}{x}$

$\frac{dy}{dx} = \frac{1}{3} - \frac{2}{x^2}$

$= \frac{x^2 - 6}{3x^2} = \frac{(x - \sqrt{6})(x + \sqrt{6})}{3x^2} > 0$



$x < -\sqrt{6}, x > \sqrt{6}$

Since $3x^2$ is always positive

$\therefore$ y is increasing for $\{x: x \le -\sqrt{6}\} \cup \{x: x \ge \sqrt{6}\}$

**4** $s = 2 - 3t + t^2$

$\frac{ds}{dt} = -3 + 2t$

$\frac{ds}{dt} < 0 \Rightarrow -3 + 2t < 0$

$t < \frac{3}{2}$

s is decreasing for $\left\{t : t \le \frac{3}{2}\right\}$

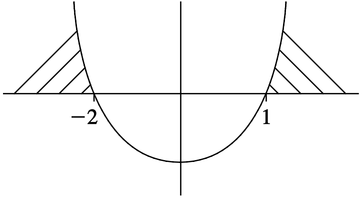
**5** $y = 2x^3 + 3x^2 - 12x + 4$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$\frac{dy}{dx} > 0 \quad \Rightarrow 6x^2 + 6x - 12 > 0$

$x^2 + x - 2 > 0$

$(x + 2)(x - 1) > 0$



$x < -2, x > 1$

y is decreasing for $\{x : x \le -2\} \cup \{x : x \ge 1\}$

**6** $y = 4x^2 + 3x + 1$

$\frac{dy}{dx} = 8x + 3$

$x = 1, \frac{dy}{dx} = 8 + 3 = 11$

$x = 1, y = 4 + 3 + 1 = 8$

Equation of tangent $y - 8 = 11(x - 1)$

$y - 8 = 11x - 11$

$y = 11x - 3$

The equation of the tangent is $y = 11x - 3$

**7** $y = \frac{4}{2x + 3}$

$x = 2, \; y = \frac{4}{2(2) + 3} = \frac{4}{7} \; \left(2, \frac{4}{7}\right)$

$\frac{dy}{dx} = \frac{-8}{(2x + 3)^2}$ using the chain rule

$x = 2, \frac{dy}{dx} = \frac{-8}{49}$

$y - \frac{4}{7} = \frac{-8}{49}(x - 2)$

$49y - 28 = -8x + 16$

$49y + 8x - 44 = 0$

Equation of the tangent $49y + 8x - 44 = 0$

**8** $y = \frac{4}{1 - 2x}$

$x = 1, \; y = \frac{4}{1 - 2} = -4$

$\frac{dy}{dx} = \frac{8}{(1 - 2x)^2}$

$x = 1, \frac{dy}{dx} = 8$

$y + 4 = 8(x - 1)$

$y = 8x - 12$

**9** $y = \frac{4}{x^2}, \quad y = 1$

$x^2 = 4 \Rightarrow x = \pm 2$

$(2, 1)\ (-2, 1)$

$\frac{dy}{dx} = \frac{-8}{x^3}$

$x = 2,\ \frac{dy}{dx} = -1$

$x = -2,\ \frac{dy}{dx} = 1$

At (2, 1), $m = -1$, $y - 1 = -(x - 2)$

$y = -x + 3$

(–2, 1) $m = 1$, $y - 1 = x + 2$

$y = x + 3$

**10** $y = x \cos x$

$\frac{dy}{dx} = \cos x - x \sin x$

$x = \frac{\pi}{2},\ y = \frac{\pi}{2}\cos\frac{\pi}{2} = 0,\ \frac{dy}{dx} = \cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}$

$y - 0 = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$

$y = \frac{-\pi}{2}x + \frac{\pi^2}{4}$

**11** $y = \sin(2x - \pi)$

$\frac{dy}{dx} = 2\cos(2x\pi -)$

$x = \pi,\ y = \sin(2\pi - \pi) = \sin\pi = 0$

$\frac{dy}{dx} = 2\cos(2\pi - \pi) = 2\cos\pi = -2$

Gradient of normal = $\frac{1}{2}$

$y - 0 = \frac{1}{2}(x - \pi)$

$y = \frac{1}{2}x - \frac{\pi}{2}$

**12** $y = x \tan x$

$\frac{dy}{dx} = \tan x + x \sec^2 x$

$x = \frac{\pi}{4},\ y = \frac{\pi}{4}\tan\frac{\pi}{4} = \frac{\pi}{4},\ \frac{dy}{dx} = \tan\frac{\pi}{4} + \frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)$

$= 1 + \frac{\pi}{2} = \frac{2 + \pi}{2}$

Gradient of normal = $\frac{-2}{2 + \pi}$

Equation of normal: $y - \frac{\pi}{4} = \frac{-2}{2 + \pi}\left(x - \frac{\pi}{4}\right)$

$(2 + \pi)\,y = -2x + \frac{\pi}{2} + (2 + \pi)\left(\frac{\pi}{4}\right)$

$(2 + \pi)y + 2x = \pi + \frac{\pi^2}{4}$

**13** $x = \frac{\pi}{2}, \quad y = \frac{2x+1}{\sin^2 x}$

$$\frac{dy}{dx} = \frac{2\sin^2 x - (2x+1)2\sin x\cos x}{\sin^4 x}$$

$$x = \frac{\pi}{2}, \quad y = \frac{\pi+1}{1} = \pi + 1, \quad \frac{dy}{dx} = \frac{2 \; - 0}{1} = 2$$

Gradient of normal = $-\frac{1}{2}$

$$y - (\pi + 1) = -\frac{1}{2}(x - \frac{\pi}{2})$$

$$y = -\frac{1}{2}x + \frac{5\pi}{4} + 1$$

**14** $y = \frac{x-2}{2x+1}$

$$\frac{dy}{dx} = \frac{(2x+1) - (x-2)(2)}{(2x+1)^2}$$

$$= \frac{5}{(2x+1)^2}$$

$y = 1, \; \frac{x-2}{2x+1} = 1$

$2x + 1 = x - 2$

$x = –3$

$$\frac{dy}{dx} = \frac{5}{25} = \frac{1}{5}$$

Gradient of normal = –5, (–3, 1)

$y - 1 = –5(x + 3)$

$y = –5x – 14$

**15** $y = (3x - 2)^{-1/2}$

$$\frac{dy}{dx} = \frac{-3}{2}(3x-2)^{-3/2}$$

$x = 1, y = 1, \; \frac{dy}{dx} = \frac{-3}{2}$

Gradient of normal $= \frac{2}{3}$

$$y - 1 = \frac{2}{3}(x - 1)$$

$$y = \frac{2}{3}x + \frac{1}{3}$$

**16** $y = x^2 – 4x + 5$

$y + 3x = 4 \Rightarrow y = –3x + 4$

$$\frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = 2x - 4 = -3$$

$2x = 1$

$$x = \frac{1}{2}$$

$x=\frac{1}{2},\ y=\left(\frac{1}{2}\right)^2-4\left(\frac{1}{2}\right)+5=3\frac{1}{4}$

$\left(1,3\frac{1}{4}\right)$ m = –3

Equation of the tangent is

$y-\frac{13}{4}=-3(x-1)$

$y=-3x+3+\frac{13}{4}$

$y=-3x+\frac{25}{4}$

**17** $y=\frac{4}{(2x-1)^2}$

$\frac{dy}{dx}=\frac{-16}{(2x-1)^3}$

$x=1,\ \frac{dy}{dx}=-16,$ Gradient of normal $=\frac{1}{16}$

$y-4=\frac{1}{16}(x-1)$

$y=\frac{1}{16}x-\frac{1}{16}+4$

$y=\frac{1}{16}x+3\frac{15}{16}$

$y=0\Rightarrow\frac{1}{16}x=-\frac{63}{16}$

x = –63

A(–63, 0)

$x=0,\ y=\frac{63}{16}$

$B\left(0,\frac{63}{16}\right)$

Length of AB = $\sqrt{(-63)^2+\left(\frac{63}{16}\right)^2}$

= 63.12

**18** $y=2x-\frac{3}{1-x}$

$\frac{dy}{dx}=2-\frac{3}{(1-x)^2}$, using the chain rule

When $x=2,\ y=2(2)-\frac{3}{1-2}=7$

$\frac{dy}{dx}=2-\frac{3}{(1-2)^2}=2-3=-1$

Gradient of the normal = 1

Equation of the normal:

y – 7 = x – 2

y = x + 5

Since $y = 2x - \frac{3}{1-x}$, if the normal meets the curve again:

$x + 5 = 2x - \frac{3}{1-x}$

$\frac{3}{1-x} = x - 5$

$3 = (x - 5)(1 - x)$

$3 = x - x^2 - 5 + 5x$

$x^2 - 6x + 8 = 0$

$(x - 2)(x - 4) = 0$

$x = 2, 4$

When $x = 4$, $y = 8 - \frac{3}{1-4} = 9$

(4, 9)

**19** (a) $y = 3 + 4x - \frac{1}{2}x^2$

$\frac{dy}{dx} = 4 - x$

$x = 2, \frac{dy}{dx} = 2$

Gradient of normal = $-\frac{1}{2}$

Equation of normal:

$y - 9 = -\frac{1}{2}(x - 2)$

$y = -\frac{1}{2}x + 10$

(b) $-\frac{1}{2}x + 10 = 3 + 4x - \frac{1}{2}x^2$

$\frac{1}{2}x^2 - 4\frac{1}{2}x + 7 = 0$

$\frac{1}{2}x^2 - \frac{9}{2}x + 7 = 0$

$x^2 - 9x + 14 = 0$

$(x - 2)(x - 7) = 0$

$x = 2, 7$

$x = 7, y = \frac{-7}{2} + 10 = \frac{13}{2}$

$A = \left(7, \frac{13}{2}\right)$

(c) $x = 7, \frac{dy}{dx} = 4 - 7 = -3$

$y - \frac{13}{2} = -3(x - 7)$

$y = -3x + 21 + \frac{13}{2}$

$y = -3x + \frac{55}{2}$

**20** $y = 4x^2\cos(2x)$

$\frac{dy}{dx} = 8x\cos 2x - 8x^2\sin(2x)$

$x = \pi, y = 4\pi^2\cos(2\pi) = 4\pi^2$

$\frac{dy}{dx} = 8\pi$

$y - 4\pi^2 = 8\pi(x - \pi)$

$y = 8\pi x - 4\pi^2$

## Exercise 14B

**1** $y = x^2 - 2x + 1$

$\frac{dy}{dx} = 2x - 2$

$\frac{d^2y}{dx^2} = 2$

$\frac{dy}{dx} = 0 \Rightarrow 2x - 2 = 0$

$x = 1$

$x = 1, y = 1 - 2 + 1 = 0$

Minimum point since $\frac{d^2y}{dx^2} > 0$

$\therefore$ (1, 0) is a minimum point

**2** $y = x^3 - 3x + 1$

$\frac{dy}{dx} = 3x^2 - 3$

$\frac{d^2y}{dx^2} = 6x$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 = 3$

$x^2 = 1$

$x = 1, -1$

When $x = 1$, $y = 1 - 3 + 1 = -1$, $\frac{d^2y}{dx^2} = 6 > 0$

$\therefore$ (1, –1) minimum point

$x = -1$, $y = -1 + 3 + 1 = 3$, $\frac{d^2y}{dx^2} = -6 < 0$

(–1, 3) maximum point

**3** $y = 2x^3 - 3x^2 - 12x + 1$

$\frac{dy}{dx} = 6x^2 - 6x - 12$

$\frac{d^2y}{dx^2} = 12x - 6$

$\frac{dy}{dx} = 0 \Rightarrow 6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2, -1$

$\frac{d^2y}{dx^2} = 0 \Rightarrow 12x - 6 = 0$

$x = \frac{1}{2}$

$x < \frac{1}{2}, \frac{dy}{dx}$ is –ve

$x > \frac{1}{2}, \frac{dy}{dx}$ is –ve

Point of inflexion at $x = \frac{1}{2}$, $y = 2\left(\frac{1}{8}\right) - 3\left(\frac{1}{4}\right) - 12\left(\frac{1}{2}\right) + 1$

$= \frac{1}{4} - \frac{3}{4} - 6 + 1$

$= -5\frac{1}{2}$

$x = 2, y = 2(2)^3 - 3(2)^2 - 12(2) + 1 = 16 - 12 - 24 + 1 = -19$

$x = -1, y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1 = -2 - 3 + 12 + 1 = 8$

$x = 2, \frac{d^2y}{dx^2} = 12(2) - 6 = 18 > 0 \Rightarrow$ minimum point

$x = -1, \frac{d^2y}{dx^2} = -12 - 6 = -18 < 0 \Rightarrow$ maximum point

(2, –19) minimum point

(–1, 8) maximum point

**4** $y = x^3 - 6x^2 + 9x - 2$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

$\frac{d^2y}{dx^2} = 6x - 12$

$\frac{dy}{dx} = 0 \quad \Rightarrow \quad 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

$x = 1, 3$

$\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 12 = 0$

$x = 2$

$x < 2, \frac{dy}{dx} < 0$

$x > 2, \frac{dy}{dx} < 0$

when $x = 2, y = 8 - 24 + 18 - 2 = 0$

(2, 0) point of inflexion

$x = 1, y = 1 - 6 + 9 - 2 = 2, \frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0$ maximum point

$x = 3$, $y = 27 - 54 + 27 - 2 = -2$, $\frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0$ minimum point

(1, 2) maximum point
(3, –2) minimum point

**5** $y = x^4 - 2x^2 + 3$

$$\frac{dy}{dx} = 4x^3 - 4x$$

$$\frac{d^2y}{dx^2} = 12x^2 - 4$$

$$\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 4x = 0$$

$4x(x^2 - 1) = 0$
$4x(x - 1)(x + 1) = 0$
$x = 0, 1, -1$

When $x = 0$, $y = 3$, $\frac{d^2y}{dx^2} = -4 < 0$ maximum point

$x=1$, $y = 1-2+3 = 2$, $\frac{d^2y}{dx^2} = 12 - 4 = 8 > 0$ minimum point

$x=-1$, $y=1-2+3=2$, $\frac{d^2y}{dx^2} = 12 - 4 = 8 > 0$ minimum point

$\therefore$ (0, 3) maximum point
(1, 2) minimum point
(–1, 2) minimum point

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 12x^2 - 4 = 0$$

$x^2 = \frac{1}{3}$, $x = \frac{1}{\sqrt{3}}$, $x = -\frac{1}{\sqrt{3}}$

When $x = \frac{1}{\sqrt{3}}$, $y = \frac{1}{9} - \frac{2}{3} + 3 = \frac{22}{9}$

When $x = -\frac{1}{\sqrt{3}}$, $y = \frac{1}{9} - \frac{2}{3} + 3 = \frac{22}{9}$

$\left(\frac{1}{\sqrt{3}}, \frac{22}{9}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{22}{9}\right)$ are points of inflexion

**6** $y = \frac{x}{x^2 + 1}$

$$\frac{dy}{dx} = \frac{x^2 + 1 - x(2x)}{(x^2 + 1)^2}$$

$$= \frac{1 - x^2}{(x^2 + 1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4}$$

$$= \frac{(-2x)(x^2 + 1) - 4x(1 - x^2)}{(x^2 + 1)^3}$$

$= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3} = \frac{2x^3 - 6x}{(x^2+1)^3}$

$\frac{dy}{dx} = 0 \Rightarrow 1 - x^2 = 0, \quad x = \pm 1$

When $x = 1$, $y = \frac{1}{2}$, $\frac{d^2y}{dx^2} = \frac{-4}{8} < 0$ maximum point

$x = -1$, $y = -\frac{1}{2}$, $\frac{d^2y}{dx^2} = \frac{4}{8} > 0$ minimum point

$\left(1, \frac{1}{2}\right)$ maximum point

$\left(-1, \frac{-1}{2}\right)$ minimum point

$\frac{d^2y}{dx^2} = 0$

$\Rightarrow$ $2x\,(x^2–3) = 0$

$x = 0$, $x = \sqrt{3}$, $-\sqrt{3}$

$x = \sqrt{3}$, $y = \frac{1}{4}\sqrt{3}$, $x < \sqrt{3}, \frac{d^2y}{dx^2} < 0 \quad x > \sqrt{3}, \frac{d^2y}{dx^2} > 0$

$(\sqrt{3}, \frac{1}{4}\sqrt{3})$ point of inflexion

$x = -\sqrt{3}$, $y = -\frac{1}{4}\sqrt{3}$, $x < -\sqrt{3}, \frac{d^2y}{dx^2} < 0 \quad x > -\sqrt{3}, \frac{d^2y}{dx^2} > 0$

$\left(-\sqrt{3}, -\frac{1}{4}\sqrt{3}\right)$point of inflexion

$x = 0$, $y = 0$, $x < 0, \frac{d^2y}{dx^2} > 0 \quad x > 0, \frac{d^2y}{dx^2} < 0$

(0, 0) point of inflexion

**7** $y = \frac{x^2 - 4}{x^2 + 4}$

$= \frac{x^2 + 4 - 8}{x^2 + 4}$

$= 1 - \frac{8}{x^2 + 4}$

$\frac{dy}{dx} = \frac{8(2x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}$

$\frac{d^2y}{dx^2} = \frac{(x^2+4)^2(16) - 16x\,2(2x)(x^2+4)}{(x^2+4)^4}$

$= \frac{16(x^2+4) - 64x^2}{(x^2+4)^3} = \frac{-48x^2 + 64}{(x^2+4)^3}$

$\frac{dy}{dx} = 0 \Rightarrow 16x = 0, x = 0$

$\frac{d^2y}{dx^2} = \frac{64}{64} = 1 > 0$ minimum point

$y = \frac{-4}{4} = -1$

$\therefore$ (0, –1) minimum point

$\frac{d^2y}{dx^2} = 0 \Rightarrow -48x^2 + 64 = 0$

$x^2 = \frac{64}{48} = \frac{4}{3}$

$x = \pm\frac{2\sqrt{3}}{3}$

$x = \frac{2\sqrt{3}}{3}, y = \frac{\frac{-8}{3}}{16/3} = \frac{-1}{2}$

$x = \frac{-2\sqrt{3}}{3},\ y = \frac{-1}{2}$

Since $\frac{dy}{dx}$ does not change sign:

x $\left(\frac{2\sqrt{3}}{3}, -\frac{1}{2}\right)$ and $\left(\frac{-2\sqrt{3}}{3}, \frac{-1}{2}\right)$ are points of inflexion

**8** Vol of cylinder = 20 $cm^3$

$A = \pi r^2 + 2\pi rh$

$V = \pi r^2 h = 20$

$h = \frac{20}{\pi r^2}$

$A = \pi r^2 + 2\pi r\left(\frac{20}{\pi r^2}\right)$

$= \pi r^2 + \frac{40}{r}$

$\frac{dA}{dr} = 2\pi r - \frac{40}{r^2}$

$\frac{d^2A}{dr^2} = 2\pi + \frac{80}{r^3}$

$\frac{dA}{dr} = 0 \Rightarrow 2\pi r - \frac{40}{r^2} = 0$

$2\pi r^3 = 40$

$r^3 = \frac{40}{2\pi}$

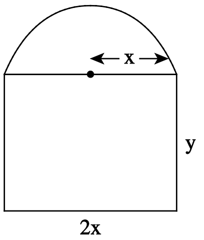
$r = \sqrt[3]{\frac{20}{\pi}}$

= 1.853

$h = \frac{20}{\pi(1.853)^2} = 1.853$

r = 1.853 cm, h = 1.853 cm

**9**



$p = 2x + 2y + \pi x.$

$15 = 2x + 2y + \pi x$

$$y = \frac{15 - 2x - \pi x}{2}$$

$$A = \frac{\pi}{2}x^2 + 2xy$$

$$= \frac{\pi}{2}x^2 + 2x\left[\frac{15 - 2x - \pi x}{2}\right]$$

$$= \frac{\pi}{2}x^2 + 15x - 2x^2 - \pi x^2$$

$$\frac{dA}{dx} = \pi x + 15 - 4x - 2\pi x$$

$= 15 - \pi x - 4x$

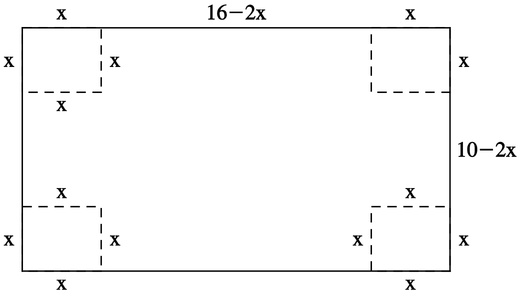
$$\frac{dA}{dx} = 0$$

$x(4+\pi) = 15$

$$x = \frac{15}{4+\pi} = 2.10$$

Width = 2(2.10) = 4.20 cm

**10**



$v = (16 - 2x)(10 - 2x)x$

$= (160 - 52x + 4x^2)x$

$= 160x - 52x^2 + 4x^3$

$$\frac{dv}{dx} = 160 - 104x + 12x^2$$

$$\frac{d^2v}{dx^2} = -104 + 24x$$

$$\frac{dv}{dx} = 0 \Rightarrow 12x^2 - 104x + 160 = 0$$

$3x^2 - 26x + 40 = 0$

$(3x - 20)(x - 2) = 0$

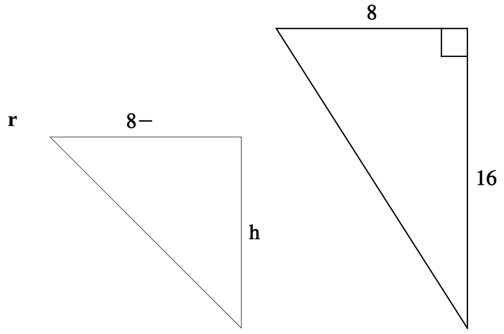
$$x = 2, \frac{20}{3}$$

$x = 2, \dfrac{d^2v}{dx} = -104 + 48 < 0$ maximum

$x = \dfrac{20}{3}, \dfrac{d^2v}{dx^2} = -104 + 160 > 0$ minimum

Maximum volume when $x = 2$, $V = 160(2) - 52(2)^2 + 4(2)^3$
$= 320 - 208 + 32$
$= 144 \text{ cm}^3$

**11** (a) $V = \dfrac{1}{3}\pi r^2 h$



$$\frac{h}{16} = \frac{8 - r}{8}$$

$8h = 16(8-r)$
$h = 2(8-r)$

(b) $V = \dfrac{1}{3}\pi r^2 (2)(8 - r)$

$= \dfrac{2}{3}\pi r^2 (8 - r)$

$V = \dfrac{16}{3}\pi r^2 - \dfrac{2}{3}\pi r^3$

$\dfrac{dv}{dr} = \dfrac{32}{3}\pi r - 2\pi r^2$

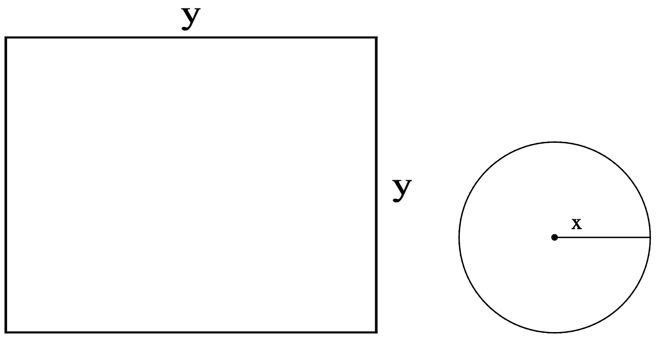
$\dfrac{dv}{dr} = 0 \Rightarrow \dfrac{32}{3}\cancel{\pi} r - 2\cancel{\pi} r^2 = 0$

$r\left(\dfrac{16}{3} - r\right) = 0 \Rightarrow r = 0, \ r = 16/3$

Since $r \neq 0 \Rightarrow r = \dfrac{16}{3}$

$V = \dfrac{16}{3}\pi\left(\dfrac{16}{3}\right)^2 - \dfrac{2}{3}\pi\left(\dfrac{16}{3}\right)^3 = \dfrac{1}{3}\pi\left(\dfrac{16}{3}\right)^3 = 158.9\text{cm}^3$

**12** (a)



$4y + 2\pi x = 120$

$2\pi x = 120 - 4y$

$x = \dfrac{120-4y}{2\pi} = \dfrac{60-2y}{\pi}$

$A = y^2 + \pi x^2$

$A = \pi\left(\dfrac{60-2y}{\pi}\right)^2 + y^2 = \dfrac{3600-240y+4y^2}{\pi} + y^2$

$A = \dfrac{4y^2+\pi y^2+3600-240y}{\pi}$

$= \dfrac{y^2(4+\pi)+3600-240y}{\pi}$

(b) $\dfrac{dA}{dy} = \dfrac{2y(4+\pi)-240}{\pi} = 0$

$y = \dfrac{240}{2(4+\pi)} = \dfrac{120}{4+\pi}$

**13** $V = 3x \times 2x \times h = 6x^2h$

$6x^2h = 144$

$h = \dfrac{144}{6x^2} = \dfrac{24}{x^2}$

(a) $A = 2(2x \times 3x) + 2(2x \times h) + 2(3x)(h)$

$= 12x^2 + 4xh + 6xh$

$= 12x^2 + 10xh$

$= 12x^2 + 10x\left(\dfrac{24}{x^2}\right)$

$= 12x^2 + \dfrac{240}{x}$

(b) (i) $\dfrac{dA}{dx} = 24x - \dfrac{240}{x^2}$

$\dfrac{dA}{dx} = 0 \Rightarrow 24x = \dfrac{240}{x^2}$

$x^3 = 10$

$x = \sqrt[3]{10} = 2.15$

(ii) $\dfrac{d^2A}{dx^2} = 24 + \dfrac{480}{x^3}$

When $x = 2.15$, $\dfrac{d^2A}{dx^2} = 24 + \dfrac{480}{2.15^3} > 0$ minimum point

$$x = 2.15,\ A = 12(2.15)^2 + \frac{240}{2.15}$$

$$= 167.1\ \text{cm}^2$$

**14** $V = x^2h$

$x^2h = 64000$

$$h = \frac{64000}{x^2}$$

$A = 4xh + x^2$

$$A = \frac{64000 \times 4}{x} + x^2$$

$$\frac{dA}{dx} = \frac{-64000 \times 4}{x^2} + 2x$$

$$\frac{dA}{dx} = 0 \Rightarrow 2x^3 = 64000 \times 4$$

$$x^3 = \frac{64000 \times 4}{2}$$

$$x = \sqrt[3]{\frac{64000 \times 4}{2}} = 50.40\ \text{cm}$$

$$\text{when } x = 50.40\ \text{cm},\ A = \frac{256000}{(50.40)} + (50.40)^2$$

$$= 7619.53\ \text{cm}^2$$

$$h = \frac{64000}{(50.4)^2} = 25.2\text{cm}$$

**Try these 14.3**

(a) $x = t^3 + 4t - 1,\ \ y = t^2 + 7t + 9$

$$\frac{dx}{dt} = 3t^2 + 4,\ \ \frac{dy}{dt} = 2t + 7$$

$$\frac{dy}{dx} = \frac{2t + 7}{3t^2 + 4}$$

$$\frac{d}{dt}\left[\frac{2t + 7}{3t^2 + 4}\right] = \frac{2(3t^2 + 4) - (2t + 7)6t}{(3t^2 + 4)^2}$$

$$= \frac{6t^2 + 4 - 12t^2 - 42t}{(3t^2 + 4)^2}$$

$$= \frac{-6t^2 - 42t + 4}{(3t^2 + 4)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-6t^2 - 42t + 4}{(3t^2 + 4)^2} \times \frac{1}{3t^2 + 4} = \frac{-6t^2 - 42t + 4}{(3t^2 + 4)^3}$$

(b) $x = \tan t,\ \ y = 2 \sin t + 1$

$$\frac{dx}{dt} = \sec^2 t,\ \ \frac{dy}{dt} = 2 \cos t$$

$$\frac{dy}{dx} = \frac{2\cos t}{\sec^2 t} = 2 \cos^3 t$$

$$t = \frac{\pi}{4}, \frac{dy}{dx} = 2\cos^3\frac{\pi}{4} = 2\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\sqrt{2}}{2}$$

$$\frac{d^2y}{dx^2} = \frac{-6\cos^2 t \sin t}{\sec^2 t} = -6\cos^4 t \sin t$$

$$t = \frac{\pi}{4}, \frac{d^2y}{dx^2} = -6\cos^4\frac{\pi}{4}\sin\frac{\pi}{4} = -6\left(\frac{\sqrt{2}}{2}\right)^4\left(\frac{\sqrt{2}}{2}\right) = \frac{-3\sqrt{2}}{4}$$

**Exercise 14C**

**1** (a) $x^3y = 10$

$\frac{dx}{dt} = 0.5$

Where $x = 5$, $y = \frac{10}{125} = \frac{2}{25}$

$y = \frac{10}{x^3}$

$\frac{dy}{dx} = \frac{-30}{x^4}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{-30}{x^4} \times 0.5$

$= \frac{-30}{5^4} \times 0.5 = -0.024$ unit per second

(b) $y = 2$, $x^3 = 5 \Rightarrow x = \sqrt[3]{5}$

$\frac{dy}{dt} = \frac{-30}{(\sqrt[3]{5})^4} \times 0.5 = -1.754$ unit per second

**2** $\frac{1}{y^2} = \frac{1}{50} - \frac{1}{x^2}$

$\frac{dx}{dt} = 5\text{cms}^{-1}$

$-2y^{-3}\frac{dx}{dy} = 2x^{-3}$

$\frac{dy}{dx} = -\frac{y^3}{x^3}$

When $x = 10$, $\frac{1}{y^2} = \frac{1}{50} - \frac{1}{100} = \frac{1}{100}$

$y = 10$

$\frac{dy}{dx} = \frac{-1000}{1000} = -1$

$\therefore \frac{dy}{dt} = -1 \times 5 = -5\text{cms}^{-1}$

**3** $\frac{dv}{dt} = 0.04\ \text{cm}^3\text{s}^{-1}$

$\frac{dA}{dt}$ when $v = 150\pi\ cm^3$

$v = \frac{4}{3}\pi r^3$

$150\cancel{\pi} = \frac{4}{3}\cancel{\pi} r^3$

$\frac{450}{4} = r^3$

$r = \sqrt[3]{\frac{450}{4}} = 4.827$ cm

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$\frac{0.05}{4\pi(4.827)^2} = \frac{dr}{dt}$

$= 0.0001708$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 8\pi(4.827) \times 0.0001708$

$= 0.02072\ cm^2 s^{-1}$

**4** $v = x^6(x^2 + 4) = x^8 + 4x^6$

$\frac{dx}{dt} = 4\,cms^{-1}$

$\frac{dv}{dx} = 8x^7 + 24x^5$

$x = 1,\ \frac{dv}{dx} = 8(1)^7 + 24(1)^5 = 32$

$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = 32 \times 4 = 128\,cm^3 s^{-1}$

**5** $\frac{dr}{dt} = 2\,cms^{-1}$

$A = \pi r^2 = 4\pi$

$r^2 = 4 \Rightarrow r = 2$ cm

$c = 2\pi r$

$\frac{dc}{dr} = 2\pi$

$\frac{dc}{dt} = \frac{dc}{dr} \times \frac{dr}{dt} = 2\pi \times 2 = 4\pi\ cms^{-1}$

**6** $\frac{dx}{dt} = 0.05\ cms^{-1}$

$v = 64 \Rightarrow x^3 = 64,\ x = 4$ cm

$A = 6x^2$

$\frac{dA}{dx} = 12x$

$$\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}=12x\times 0.05$$

$= 12 \times 4 \times 0.05 = 2.4\ \text{cm}^2\text{s}^{-1}$

**7** $\frac{dc}{dt}=4\ \text{cms}^{-1}$

(a) $c = 2\pi r$

$$\frac{dc}{dt}=2\pi$$

$$\frac{dc}{dt}=\frac{dc}{dr}\times\frac{dr}{dt}$$

$$4 = 2\pi\times\frac{dr}{dt}$$

$$\frac{2}{\pi}=\frac{dr}{dt}$$

(b) $\frac{dA}{dt}$ when $r = 64$ cm

$A = \pi r^2$

$$\frac{dA}{dr}=2\pi r$$

$$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$$

$$=2\pi(64)\times\frac{2}{\pi}$$

$= 256\ \text{cm}^2\text{s}^{-1}$

**8** (a) $\frac{dA}{dt}=10\ \text{cm}^2\text{s}^{-1}$

$A = 4\pi r^2$

$$\frac{dA}{dr}=8\pi r$$

$$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$$

$$10 = 8\pi r\times\frac{dr}{dt}$$

$$\frac{dr}{dt}=\frac{10}{8\pi r}$$

(b) $v = \frac{4}{3}\pi r^3$

$$\frac{dv}{dr}=4\pi r^2$$

$$\frac{dv}{dt}=\frac{dv}{dr}\times\frac{dr}{dt}$$

$$=4\pi(4^2)\times\frac{10}{8\pi(4)}=20\ \text{cm}^3\text{s}^{-1}$$

**9** $y = \frac{4x+2}{x+1}$

$$\frac{dy}{dx}=\frac{(x+1)(4)-(4x+2)}{(x+1)^2}$$

$$= \frac{4x+4-4x-2}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2}$$

y = 5

$$\frac{4x+2}{x+1} = 5$$

4x + 2 = 5x + 5

x = –3.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$0.3 = \frac{2}{(-3+1)^2} \times \frac{dx}{dt}$$

$$0.6 = \frac{dx}{dt}$$

**10** $x = 4,\ \frac{dv}{dt} = 0.024\ \text{cm}^3\text{s}^{-1}$

$v = x^3$

$$\frac{dv}{dx} = 3x^2,\ x = 4,\ \frac{dv}{dx} = 3(4)^2 = 48$$

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$0.024 = 48\ \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{0.024}{48} = 0.0005\ \text{cms}^{-1}$$

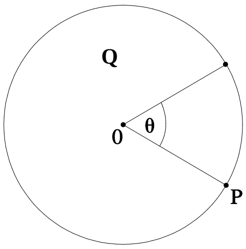
**11** (a) pv = 600

$$p = \frac{600}{v}$$

$$\frac{dp}{dv} = \frac{-600}{v^2}$$

(b) $v = 20,\ \frac{dp}{dv} = \frac{-600}{20^2} = -1.5$

**12** (a)



$$\frac{d\theta}{dt} = \frac{\pi}{3}\ \text{rad per sec}$$

$$S = r\theta = 6\theta$$

$$\frac{ds}{d\theta} = 6$$

$$\frac{ds}{dt} = \frac{ds}{d\theta} \times \frac{d\theta}{dt}$$

$= 6\ (\pi/3) = 2\pi$ cm sec

(b) $A = \frac{1}{2}r^2\theta$

$\frac{dA}{d\theta} = \frac{1}{2}r^2$

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= \frac{1}{2}(6)^2 \times \frac{\pi}{3}$

$= 6\pi\ \text{cm}^2\text{s}^{-1}$

**13** (a) $\frac{dv}{dt} = 16\text{cm}^3\text{s}^{-1}$

$v = 2x^2 - 7x$

$v = 4 \Rightarrow 2x^2 - 7x - 4 = 0$

$(2x + 1)(x - 4) = 0$

$x = -\frac{1}{2}, 4$

$x = 4$

(b) $\frac{dv}{dx} = 4x - 7$

$x = 4,\ \frac{dv}{dx} = 8 - 7 = 1$

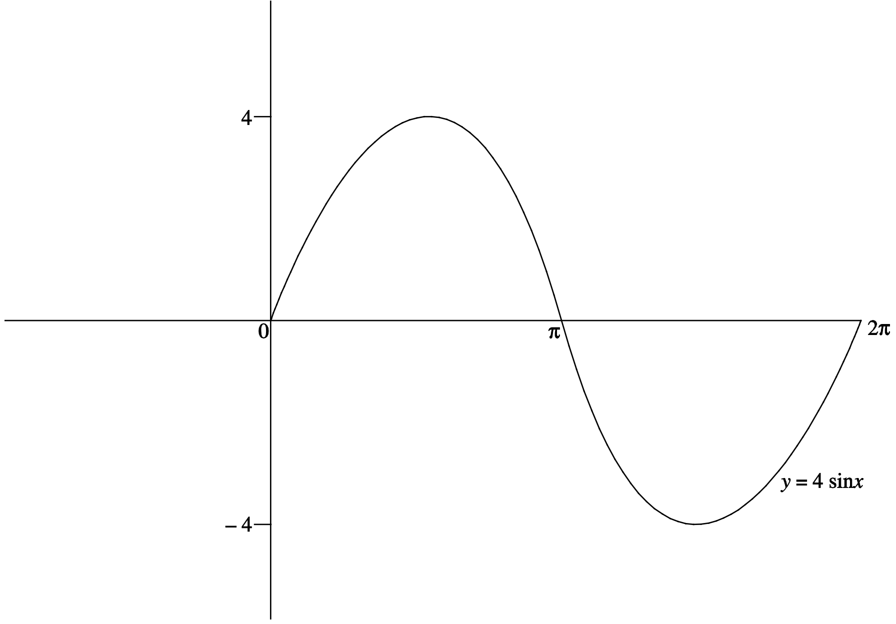
$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$16 = \frac{dx}{dt}$

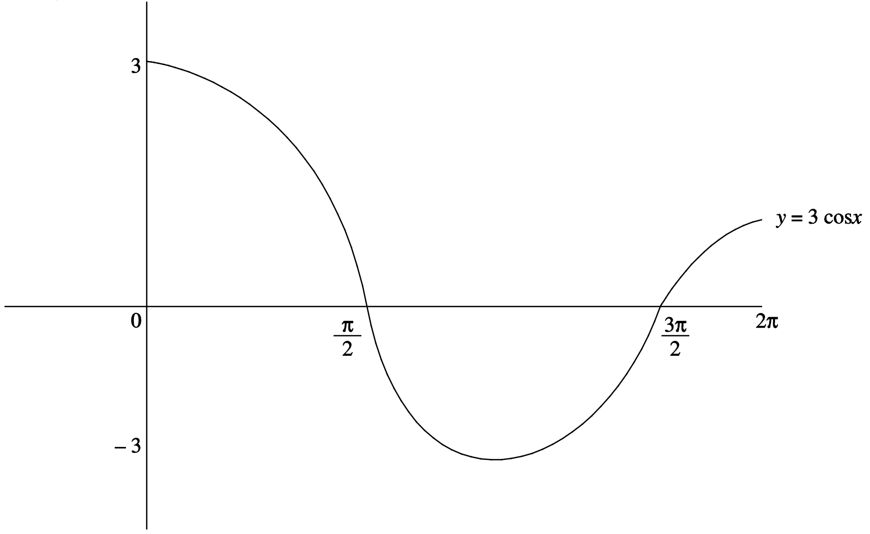
16 $\text{cms}^{-1}$

**Try these 14.4**

(a) y = 4 sin x, period = 2π, amplitude = 4



(b) y = 3 cosx, period = 2π, amplitude = 3



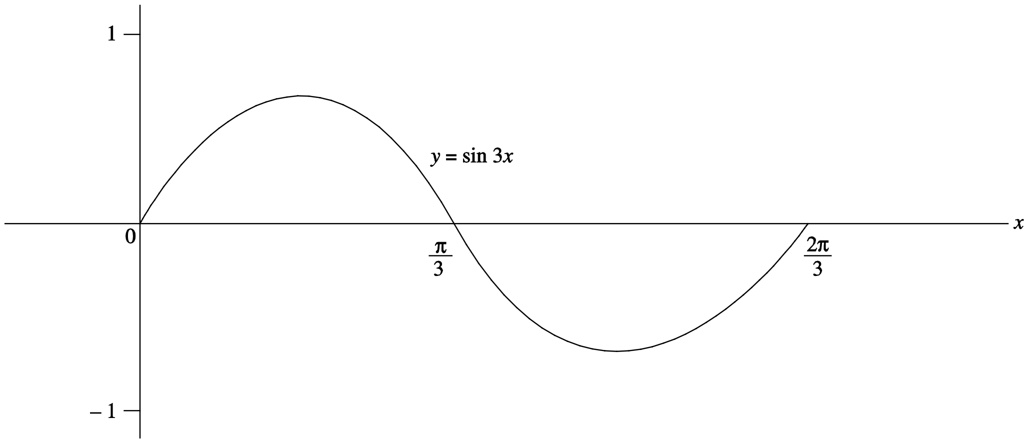
**Try these 14.5**

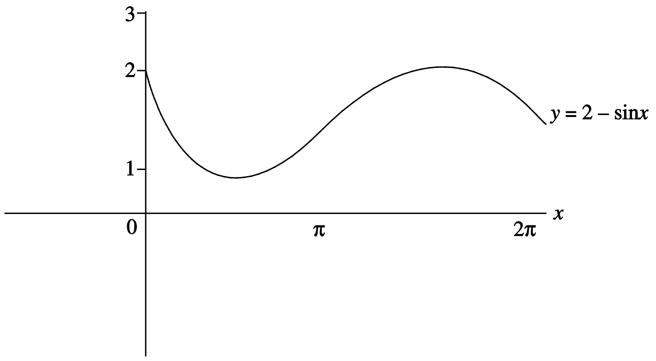
(a) $y = \cos\left(3x - \frac{\pi}{12}\right)$ ,period = $\frac{2\pi}{3}$, amplitude = 1, displacement = $\frac{\frac{\pi}{12}}{3} = \frac{\pi}{36}$

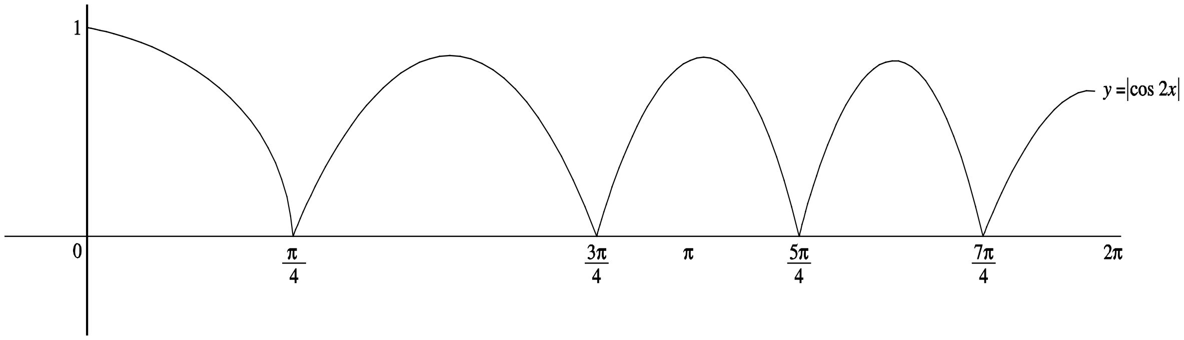
(b) $y = 40\cos\left(2\pi x - \frac{\pi}{8}\right)$ ,period = $\frac{2\pi}{2\pi} = 1$, amplitude = 2, displacement = $\frac{\frac{\pi}{8}}{2\pi} = \frac{1}{16}$

(c) $y = 2\sin\left(x - \frac{\pi}{3}\right)$ ,period = $2\pi$, amplitude = 2, displacement = $\frac{\pi}{3}$

**Try these 14.6**







**Try these 14.7**

(a) $y = \dfrac{2x+3}{4x-1}$

Horizontal asymptote: $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{4x-1}\right) = \lim\limits_{x\to\infty}\left(\dfrac{2+\frac{3}{x}}{4-\frac{1}{x}}\right) = \dfrac{2}{4} = \dfrac{1}{2}$

$y = \dfrac{1}{2}$

Vertical asymptote: $4x – 1 = 0$

$x = \dfrac{1}{4}$

$\therefore$ Horizontal Asymptote: $y = \dfrac{1}{2}$

Vertical Asymptote: $x = \dfrac{1}{4}$

(b) $y = \dfrac{x+1}{x-3}$

$\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-3}\right) = \lim\limits_{x\to\infty}\left(\dfrac{1+\frac{1}{x}}{1-\frac{3}{x}}\right) = 1$

Horizontal Asymptote $y = 1$
Vertical Asymptote $x = 3$

(c) $y = \dfrac{x^2+2x}{x^2-7x+12}$

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\frac{x^2+2x}{x^2-7x+12}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1+\frac{2}{x}}{1-\frac{7}{x}+\frac{12}{x^2}}\right)$$

= 1

∴ y = 1 is a Horizontal asymptote

Vertical asymptote: $x^2 - 7x + 12 = 0$

(x – 3)(x – 4) = 0

x = 3, 4

∴ Vertical asymptote: x = 3, x = 4

**Exercise 14D**

**1** $y = x^2 + 2x + 1$

$$\frac{dy}{dx} = 2x + 2$$

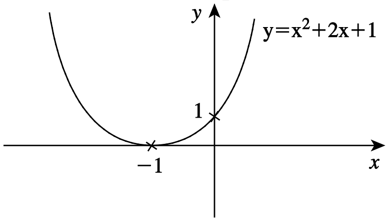
$$\frac{d^2y}{dx^2} = 2$$

$$\frac{dy}{dx} = 0 \Rightarrow 2x + 2 = 0$$

x = –1

when x = –1, y = 1 – 2 + 1 = 0.

∴ (–1, 0) is a min point



**2** $y = 12x - x^3$

$$\frac{dy}{dx} = 12 - 3x^2$$

$$\frac{d^2y}{dx^2} = -6x$$

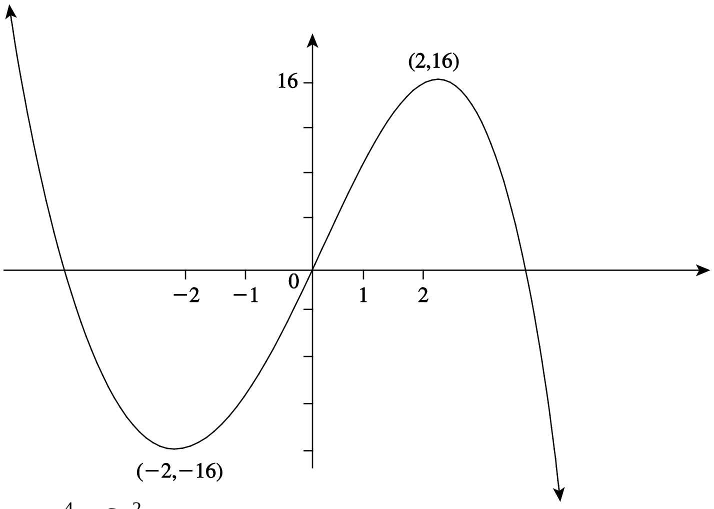
$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad 12 - 3x^2 = 0$$

$x^2 = 4$

x = ±2

when x = 2, y = 12(2) – $(2)^3$ = 16, $\frac{d^2y}{dx^2}$ – ve Maximum point

x = –2, y = 12(–2) – $(–2)^3$ = –16, $\frac{d^2y}{dx^2}$ +ve min point



**3** $y = x^4 - 6x^2$

$\frac{dy}{dx} = 4x^3 - 12x$

$\frac{d^2y}{dx^2} = 12x^2 - 12$

When $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$x = 0, x = \pm\sqrt{3}$

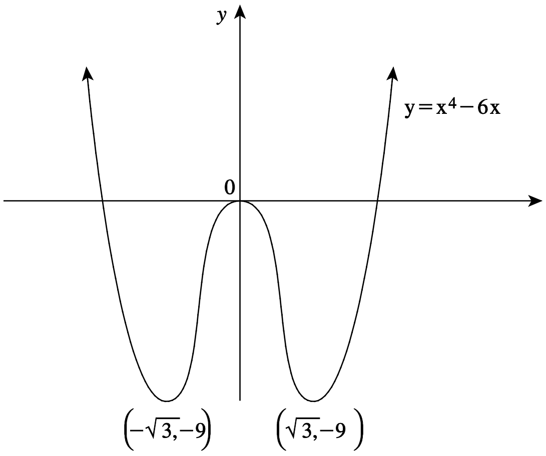
When $x = 0$, $y = 0$, $\frac{d^2y}{dx^2} = 12(0)^2 - 12 = -12 < 0 \Rightarrow$ Max. pt at $(0,0)$

$x = \sqrt{3}$, $y = \left(\sqrt{3}\right)^4 - 6\left(\sqrt{3}\right)^2 = 9 - 18 = -9$, $\frac{d^2y}{dx^2} = 12\left(\sqrt{3}\right)^2 - 12 = 24 > 0$

Minimum point at $(\sqrt{3}, -9)$

$x = -\sqrt{3}$, $y = \left(-\sqrt{3}\right)^4 - 6\left(-\sqrt{3}\right)^2 = -9$, $\frac{d^2y}{dx^2} = 12\left(-\sqrt{3}\right)^2 - 12 = 24 > 0$

Minimum point at $(-\sqrt{3}, -9)$

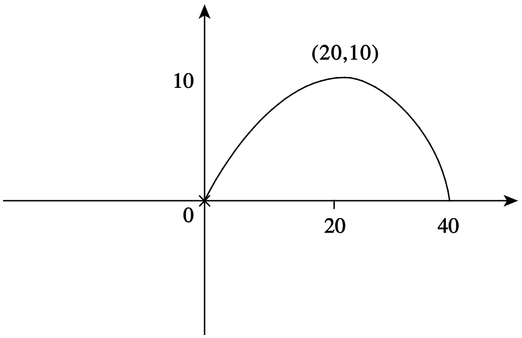


**4** $y = \frac{-1}{40}x^2 + x$

When $y = 0$, $x\left(\frac{-1}{40}x + 1\right) = 0$, $x = 0, 40$

$$\frac{dy}{dx}=\frac{-1}{20}x+1$$

When $\frac{dy}{dx}=0$, $x = 20$, $y = 10$



**5** $\theta = -2t^3 + 12t^2 + 10$

$$\frac{d\theta}{dt}=-6t^2+24t$$

$$\frac{d^2\theta}{dt^2}=-12t+24$$

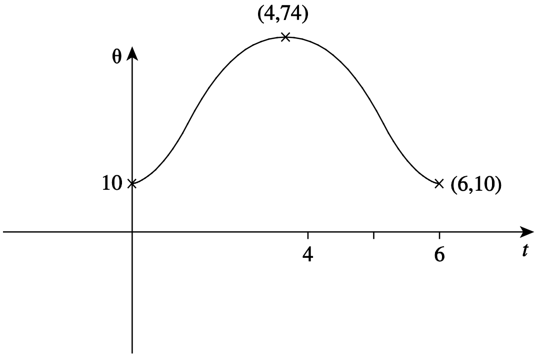
$$\frac{d\theta}{dt}=0\Rightarrow -6t^2+24t=0$$

$6t(-t + 4) = 0$

$t = 0, 4$

$t = 0$, $\theta = 10$, $\frac{d^2\theta}{dt^2} = 24 > 0$ minimum point

$t = 4$, $\theta = -2(4)^3 + 12(4)^2 + 10 = 74$. $\frac{d^2\theta}{dt^2}$ $-24 < 0$ maximum point



When $t = 6$, $\theta = -2(6)^3+12(6)^2+10=10$

**6** (a) $y = \frac{2x+1}{x-3}$

$$\frac{dy}{dx}=\frac{(x-3)(2)-(2x+1)}{(x-3)^2}$$

$$=\frac{-7}{(x-3)^2}$$

$$\frac{dy}{dx}=0\Rightarrow -7=0\Rightarrow \text{inconsistent}$$

$\Rightarrow$ y has no turning points

(b) Horizontal asymptotes when $x - 3 = 0$

$x = 3$

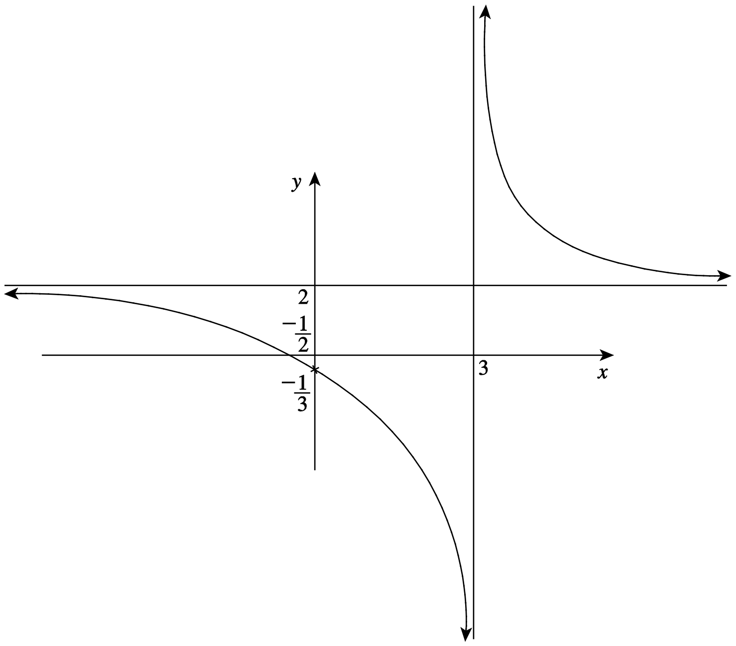
Vertical asymptotes when $\lim_{x\to\infty}\frac{2x+1}{x-3} = 2$

$y = 2$

Asymptotes $x = 3$, $y = 2$

(c) when $x = 0$, $y = -\frac{1}{3}$

$y = 0$, $x = -\frac{1}{2}$



**7** $y = \frac{2}{x^2+4}$

$x = 0$, $y = \frac{1}{2}$

$\frac{dy}{dx} = \frac{-4x}{(x^2+4)^2}$

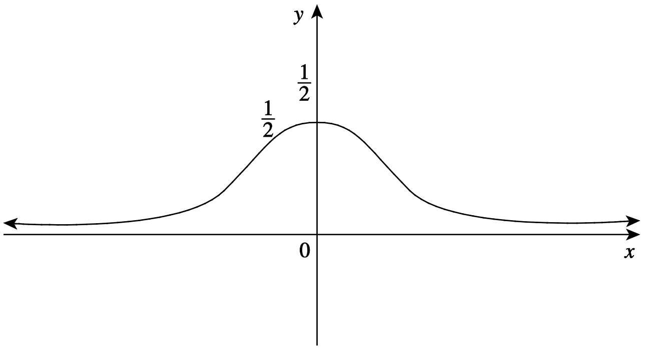
$\frac{dy}{dx} = 0 \Rightarrow x = 0$

$x < 0$, $\frac{dy}{dx} + \text{ve}$

$x > 0$, $\frac{dy}{dx} - \text{ve}$

$x = 0$, $y = \frac{1}{2}$ : Maximum point

$x \to \pm\infty$, $y \to 0$



**8** $y = x + \frac{4}{x}$

$\frac{dy}{dx} = 1 - \frac{4}{x^2}$

$\frac{d^2y}{dx^2} = \frac{8}{x^3}$

$\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{4}{x^2} = 0$

$x^2 = 4$

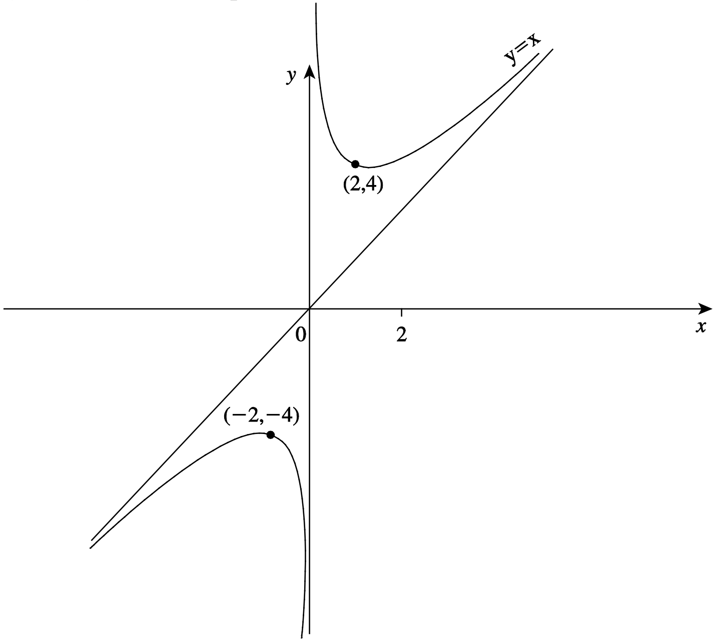
$x = \pm 2$

When $x = 2$, $y = 2 + \frac{4}{2} = 4$, $\frac{d^2y}{dx^2} = \frac{8}{8} > 0$ Minimum point

$x = –2$, $y = –2–2 = –4$, $\frac{d^2y}{dx^2} = \frac{8}{–8} < 0$ Max point

(2, 4) minimum point

(–2, –4) maximum point



Asymptotes

$y = x$

$x = 0$

**9** $y = \frac{2x+1}{x-2}$

$x = 0, y = -\frac{1}{2}$ $(0, -\frac{1}{2})$

$\frac{dy}{dx} = \frac{(x-2)(2)-(2x+1)}{(x-2)^2}$

$= \frac{-5}{(x-2)^2}$

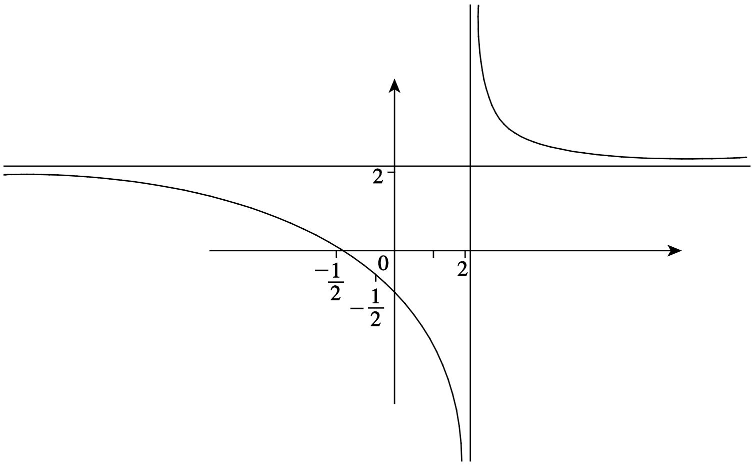
$\frac{dy}{dx} = 0 \Rightarrow -5 = 0$ Inconsistent, no turning points

Asymptotes: $x = 2$

$y = 2$

when $x = 0, y = -\frac{1}{2}$

$y = 0, x = -\frac{1}{2}$



**10** $y = \frac{x^2+2x+5}{x+2}$

$x + 2 \overline{\smash{)}\, x^2+2x+5}$ with quotient $x$

$\frac{x^2+2x}{5}$

$\therefore\ y = x + \frac{5}{x+2}$

$x = 0, y = \frac{5}{2}$

$\left(0, \frac{5}{2}\right)$

$\frac{dy}{dx} = 1 - 5(x + 2)^2$ Asymptotes

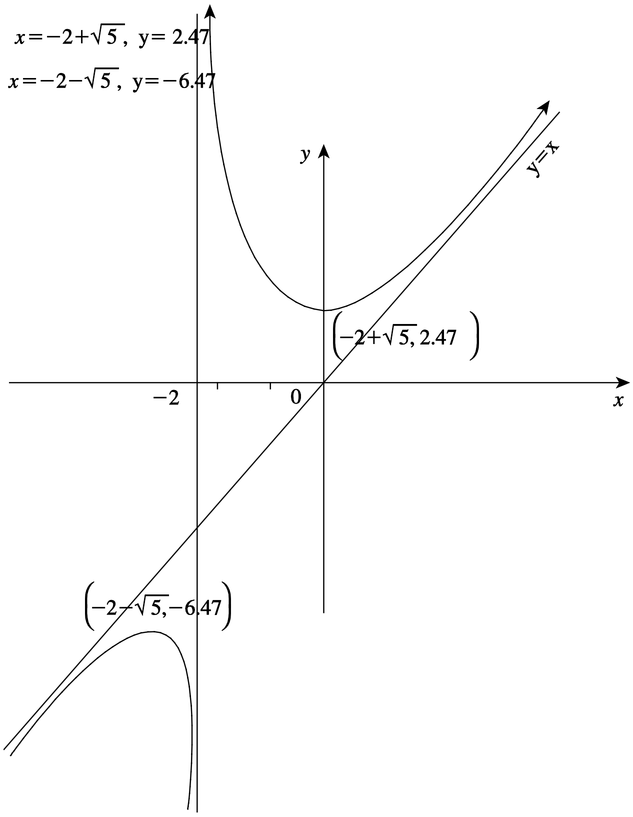
$\frac{d^2y}{dx^2} = \frac{10}{(x+2)^3}$ $x = -2$

$\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{5}{(x+2)^2} = 0$ $y = x$

$(x + 2)^2 = 5$

$x + 2 = \pm\sqrt{5}$

$x = -2 \pm \sqrt{5}$



## Review exercise 14

**1** $y = \dfrac{x}{1-x^3}$

$$\frac{dy}{dx} = \frac{(1-x^3) - x(-3x^2)}{(1-x^3)^2}$$

$$= \frac{1-x^3+3x^3}{(1-x^3)^2}$$

$$= \frac{1+2x^3}{(1-x^3)^2}$$

$$x = 2,\ y = \frac{2}{1-2^3} = \frac{2}{-7}$$

$$\frac{dy}{dx} = \frac{1+2(2)^3}{(1-2^3)2} = \frac{17}{49}$$

Gradient of normal = $-\dfrac{49}{17}$

Equation of the normal:

$$y + \frac{2}{7} = \frac{17}{49}(x-2)$$

$$y = \frac{17}{49}x + \frac{34}{49} - \frac{2}{7}$$

$$y = \frac{17}{49}x + \frac{20}{49}$$

**2** (a) $\frac{dv}{dt} = 40\ \text{cm}^3\text{s}^{-1}$

$v = 0.02\ h^3 + 0.4\ h^2 + 400\ h$

$\frac{dv}{dh} = 0.06h^2 + 0.8h + 400$

$h = 10,\ \frac{dv}{dh} = 0.06(10)^2 + 0.8(10) + 400 = 414$

$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

$40 = 414\ \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{40}{414} = 0.097\ \text{cms}^{-1}$

(b) $\frac{dh}{dt} = 0.04$

$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

$40 = 0.04\ [0.06\ h^2 + 0.8\ h + 400]$

$1000 = 0.06\ h^2 + 0.8\ h + 400$

$0.06\ h^2 + 0.8\ h - 600 = 0$

$$h = \frac{-0.8 \pm \sqrt{(0.8)^2 + 4(0.06)(600)}}{2(0.06)}$$

$$= \frac{-0.8 \pm 12.027}{0.12}$$

$h = 93.6$ cm

**3** $y = \cos x + \sin x$

$\frac{dy}{dx} = -\sin x + \cos x$

$y + x = 3$

$y = -x + 3$

Gradient = –1

Gradient of tangent = 1

$\therefore -\sin x + \cos x = 1$

$\cos x - \sin x = R\cos(x + \alpha)$

$= R\cos x \cos\alpha - R\sin x \sin\alpha$

$R\cos\alpha = 1$

$R\sin\alpha = 1$

$\tan\alpha = 1 \Rightarrow \alpha = \pi/4$

$R = \sqrt{2}$

$\sqrt{2}\ \cos(x + \pi/4) = 1$

$\cos(x + \pi/4) = \frac{1}{\sqrt{2}}$

$x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}$

$x = 0, \dfrac{6\pi}{4}$

$y = \cos 0 + \sin 0 = 1$

$\therefore$ (0, 1) → coordinate

**4** (a) $y = \dfrac{x+1}{2x-7}$

$\dfrac{dy}{dx} = \dfrac{(2x-7)-(x+1)(2)}{(2x-7)^2}$

$= \dfrac{-9}{(2x-7)^2}$

$x = 3, \dfrac{dy}{dx} = \dfrac{-9}{1} = -9$

Equation of tangent $y + 4 = -9(x - 3)$

$y = -9x + 27 - 4$

$y = -9x + 23$

(b) Gradient of normal = $\dfrac{1}{9}$

Equation of normal:

$y + 4 = \dfrac{1}{9}(x - 3)$

$y = \dfrac{1}{9}x - \dfrac{1}{3} - 4$

$y = \dfrac{1}{9}x - \dfrac{13}{3}$

(c) $y = 0, x = \dfrac{23}{9}$

$A\left(\dfrac{23}{9}, 0\right)$

$x = 0, y = \dfrac{-13}{3}$ $B\left(0, \dfrac{-13}{3}\right)$

$\text{Area of } \Delta = \dfrac{23}{9} \times \dfrac{\frac{13}{3}}{2} = 5.54$

**5** $r = 4\text{ m}, \dfrac{dA}{dt} = 3\text{ m}^2\text{s}^{-1}$

$v = \dfrac{4}{3}\pi r^3, \quad A = 4\pi r^2$

$\dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt}$

$\dfrac{dA}{dr} = 8\pi r$

$3 = 8\pi r \times \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{3}{8\pi(4)} = \dfrac{3}{32\pi} = 0.03\text{ ms}^{-1}$

$\frac{dv}{dr} = 4\pi r^2$

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$= 4\pi(4)^2 \times \frac{3}{32\pi}$

$= 6\ m^3s^{-1}$

**6** Area of cross section = 20 h cm$^2$

A = 20 h

$\frac{dA}{dt} = 0.05$

$\frac{dA}{dh} = 20$

$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

$0.05 = 20\ \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{0.05}{20} = 0.0025\ cms^{-1}$

V = 20h × h = 20h$^2$

$\frac{dv}{dh} = 40h$

$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

= 40h × 0.0025

= 40 (4) (0.0025)

$= \frac{2}{5} cm^3s^{-1}$

**7** (a) $y = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x + 1$

$\frac{dy}{dx} = x^3 - 6x^2 + 11x - 6$

$\frac{d^2y}{dx^2} = 3x^2 - 12x + 11$

For stationary points $\frac{dy}{dx} = 0$

$x^3 - 6x^2 + 11x - 6 = 0$

$x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$

$= (x - 1)(x - 2)(x - 3)$

$= 0$

x = 1, 2, 3

(b) $x = 1, y = \frac{1}{4} - 2 + \frac{11}{2} - 6 + 1 = \frac{-5}{4}.,\ \frac{d^2y}{dx^2} = 3 - 12 + 11 > 0$ minimum point

$x = 2, y = 4 - 16 + 22 - 12 + 1 = -1,\ \frac{d^2y}{dx^2} = 12 - 24 + 11 < 0$ maximum point

$x = 3, y = \frac{81}{4} - 54 + \frac{99}{2} - 18 + 1 = \frac{-5}{4},\ \frac{d^2y}{dx^2} = 27 - 36 + 11 > 0$ Min pt.

(c) Inflexion points $\frac{d^2y}{dx^2} = 0 \Rightarrow 3x^2 - 12x + 11 = 0$

$x = \frac{12 \pm \sqrt{144 - 132}}{6}$

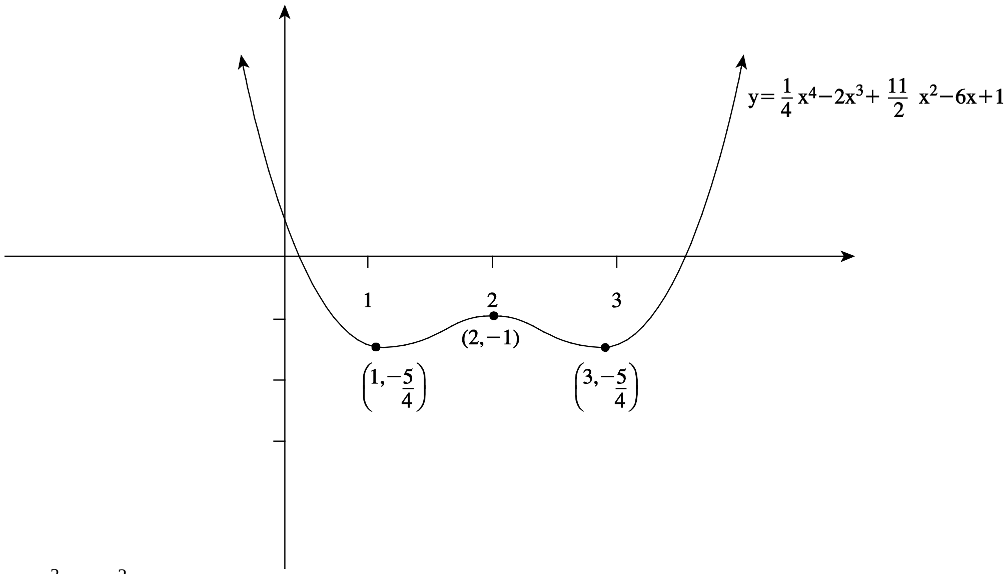
$= \frac{12 \pm \sqrt{12}}{6} = 2 \pm \frac{1}{3}\sqrt{3}$

$= 2.58, 1.42$

Points of inflexion at $x = 2.58$, $y = -1.14$

$x = 1.42$, $y = -1.14$

(d)



**8** (a) $y = x^3 + 3x^2 - 24x + 10$

$\frac{dy}{dx} = 3x^2 + 6x - 24$

$\frac{d^2y}{dx^2} = 6x + 6$

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 6x - 24 = 0$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = -4, 2$

$x = -4$, $\frac{d^2y}{dx^2} = -24 + 6 < 0$ maximum point

$x = 2$, $\frac{d^2y}{dx^2} = 12 + 6 > 0$ minimum point

Point of inflexion

$\frac{d^2y}{dx^2} = 0$, $6x + 6 = 0$

$x = -1$

$x = -1$, $y = -1 + 3 + 24 + 10 = 36$

Point of inflexion $(-1, 36)$

(b) $y = \frac{\cos 2x}{1 + \sin 2x}$

$$\frac{dy}{dx} = \frac{(1+\sin 2x)(-2\sin 2x) - (\cos 2x)(2\cos 2x)}{(1+\sin 2x)^2}$$

$$= \frac{-2\sin 2x - 2[\sin^2 2x + \cos^2 2x]}{(1+\sin 2x)^2}$$

$$= \frac{-2}{1+\sin 2x}$$

$$x = \pi/2, \ \frac{dy}{dx} = \frac{-2}{1+\sin \pi} = -2, \ y = \frac{\cos \pi}{1+\sin \pi} = -1$$

Gradient of normal = $\frac{1}{2}$, $(\pi/2, -1)$

$$y + 1 = \frac{1}{2}(x - \pi/2)$$

$$y + 1 = \frac{1}{2}x - \pi/4$$

$$y = 0 \Rightarrow 1 = \frac{1}{2}x - \pi/4$$

$$1 + \pi/4 = \frac{1}{2}x$$

$x = 2 + \pi/2$

p $(2 + \pi/2, 0)$

**9** (a) $V = yx^2$

$A = x^2 + 4xy$

$96 = x^2 + 4xy$

$$y = \frac{96 - x^2}{4x}$$

$$V = x^2\left[\frac{96 - x^2}{4x}\right]$$

$$= 24x - \frac{1}{4}x^3$$

(b) $\frac{dV}{dx} = 24 - \frac{3}{4}x^2$

$$\frac{dV}{dx} = 0 \Rightarrow \quad x^2 = \frac{24 \times 4}{3}$$

$x^2 = 32$

$x = \sqrt{32}$

$$V = 24\sqrt{32} - \frac{1}{4}\left(\sqrt{32}\right)^3$$

$$= 24\sqrt{32} - 8\sqrt{32} = 16\sqrt{32} = 64\sqrt{2}$$

**10** (a) $y = 4x^3 - 24x^2 + 36x$

$$\frac{dy}{dx} = 12x^2 - 48x + 36$$

$$\frac{d^2y}{dx^2} = 24x - 48$$

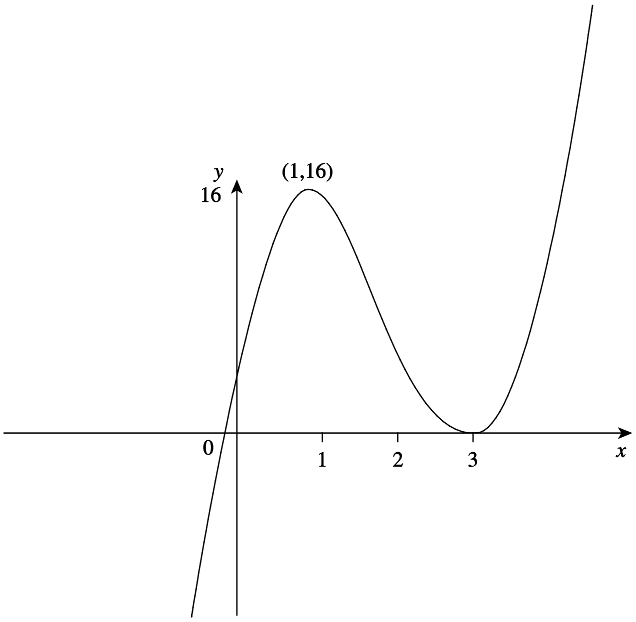
$$\frac{dy}{dx} = 0 \Rightarrow 12x^2 - 48x + 36 = 0$$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$
$x = 1, 3$
$x = 1, y = 4 - 24 + 36 = 16$
$x = 3, y = 4(3)^3 - 24(3)^2 + 36(3) = 0$
$(1, 16)\ (3, 0)$

(b) $\dfrac{d^2y}{dx^2} = 0$
$24x - 48 = 0$
$x = 2$
$y = 4(2)^3 - 24(2)^2 + 36(2) = 8$
Point of inflexion (2,8)

(c)



Where $x = 1$, $\dfrac{d^2y}{dx^2} = 24 - 48 < 0$ maximum point

$x = 3$, $\dfrac{d^2y}{dx^2}$ $24(3) - 48 > 0$ minimum point

**11** (a) $A = 243\pi$
$\pi r^2 + 2\pi rh = 243\pi$
$h = \dfrac{243 - r^2}{2r}$
$V = \pi r^2 h.$
$= \pi r^2 \left(\dfrac{243 - r^2}{2r}\right)$

$= \dfrac{\pi}{2} r\,(243 - r^2)$

$= \dfrac{\pi}{2}(243r - r^3)$

(b) $\dfrac{dv}{dr} = \dfrac{\pi}{2}(243 - 3r^2)$
$\dfrac{dv}{dr} = 0 \Rightarrow 243 = 3r^2$

$r^2 = 81$
$r = 9$
$r = 9,\ V = \frac{\pi}{2}(243(9) - 9^3)$
$= 729\pi$
$\frac{d^2v}{dr^2} = \frac{\pi}{2}(-6r)$
$r = 9,\ \frac{d^2v}{dr^2} = -27\pi < 0$ maximum point

**12** $\frac{dx}{dt} = 0.25$

(a) $x = 1,\ y = \frac{4}{(3)^2} = \frac{4}{9}$
$\frac{dy}{dx} = \frac{-40}{(5x-2)^3}$
$x = 1,\ \frac{dy}{dx} = \frac{-40}{27}$
$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{-40}{27} \times 0.25$
$= \frac{-10}{27}$

(b) $y = 9,\ \frac{4}{(5x-2)^2} = 9$
$(5x-2)^2 = \frac{4}{9}$
$5x - 2 = 2/3$
$5x = \frac{8}{3}$
$x = 8/15$
$\frac{dy}{dx} = \frac{-40}{\left(5\left(\frac{8}{15}\right) - 2\right)^3} = -135$
$\frac{dy}{dt} = -135 \times \frac{1}{4}$
$= \frac{-135}{4}$

**13** (a) $y = x^3 + 3x^2 + 3x + 2$
$\frac{dy}{dx} = 3x^2 + 6x + 3$
$3x^2 + 6x + 3 = 0$
$x^2 + 2x + 1 = 0$
$(x + 1)^2 = 0$
$x = -1$
$x = -1,\ y = -1 + 3 - 3 + 2 = 1$
$(-1, 1)$
$\frac{d^2y}{dx^2} = 6x + 6 = 0$

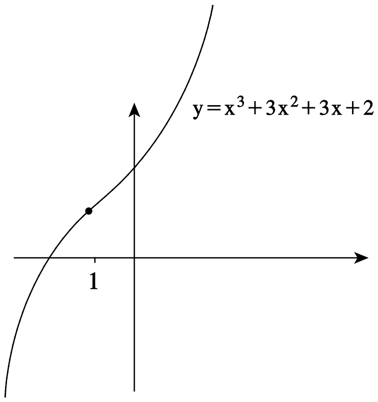
$x = -1$

$x < -1,\ \dfrac{dy}{dx} > 0$

$x > -1\ \dfrac{dy}{dx} > 0$

$x = -1$ point of inflexion

(b)



**14** $P = \dfrac{36R}{R^2 + 2R + 1}$. R = 0, P = 0

$$\frac{dP}{dR} = \frac{(R^2 + 2R + 1)(36) - 36R(2R + 2)}{(R^2 + 2R + 1)^2}$$

$$= \frac{36R^2 + 72R + 36 - 72R^2 - 72R}{(R^2 + 2R + 1)^2}$$

$$= \frac{-36R^2 + 36}{(R^2 + 2R + 1)^2}$$

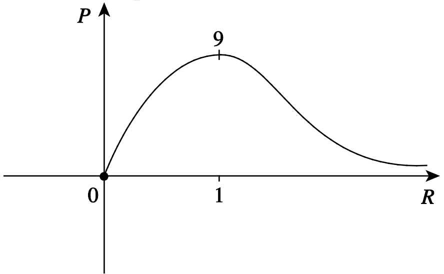
$$\frac{dP}{dR} = 0 \quad \Rightarrow 36 - 36R^2 = 0$$

$R^2 = 1$

$R = \pm 1$

When R = 1, $P = \dfrac{36}{1 + 2 + 1} = 9$

Maximum point at (1, 9)



**15** $y = \cos^3 2x + \sin^4 2x$

$x = \pi/2$, $y = \cos^3\pi + \sin^4\pi = -1$

$$\frac{dy}{dx} = -6\cos^2(2x)\sin(2x) + 8\sin^3(2x)\cos(2x)$$

$x = \pi/2$, $\dfrac{dy}{dx} = -6\cos^2\pi\ \sin\pi\ + 8\sin^3\pi\ \cos\pi\ = 0$

Equation of tangent:

$y - (-1) = 0(x - \pi/2)$

$y + 1 = 0$
$y = -1$

**16** $y = \dfrac{9x}{x^2+9}$

$x = 0, y = 0$

$\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \dfrac{4x}{x^2+9} = 0$

$\therefore$ y = 0 is an asymptote

$\dfrac{dy}{dx} = \dfrac{(x^2+9)(9) - 9x(2x)}{(x^2+9)^2}$

$= \dfrac{9x^2 + 81 - 18x^2}{(x^2+9)^2}$

$= \dfrac{-9x^2 + 81}{(x^2+9)^2}$

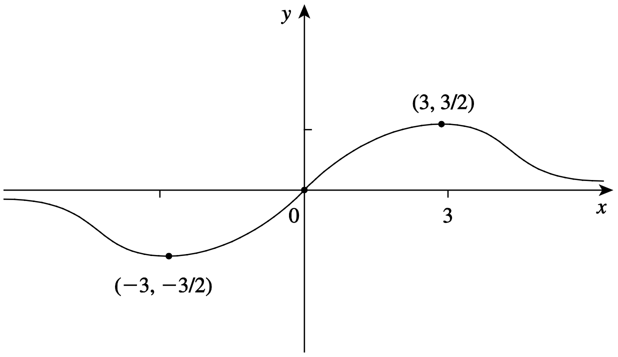
$\dfrac{dy}{dx} = 0 \Rightarrow \quad 9x^2 = 81$

$x^2 = 9$

$x = \pm 3$

when $x = 3$, $y = \dfrac{27}{18} = \dfrac{3}{2}$

$x = -3$, $y = \dfrac{-27}{18} = -3/2$



**17** $V = \dfrac{1}{3}\pi r^2 h$

$\dfrac{16}{r} = \dfrac{24}{x}$

$r = \dfrac{16x}{24} = \dfrac{2}{3}x$

$V = \dfrac{1}{3}\pi\left(\dfrac{2}{3}x\right)^2 x$

$= \dfrac{1}{3}\pi \times \dfrac{4}{9}x^3$

$= \dfrac{4}{27}\pi x^3$

$\dfrac{dv}{dt} = 0.05 \text{ cm}^3\text{s}^{-1}$

$$\frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}$$

$$\frac{dv}{dx}=\frac{4}{9}\pi x^2$$

When $x = 12$, $\frac{dv}{dx}=\frac{4}{9}\pi(12)^2$

$= 64\,\pi$

$0.05 = 64\,\pi\times\frac{dx}{dt}$

$$\frac{dx}{dt}=\frac{0.05}{64\pi}=0.00025\ \text{cms}^{-1}$$

**18** Surface area = 2400 $\text{cm}^2$

$\Rightarrow x^2 + 4xh = 2400$

$$h=\frac{2400-x^2}{4x}$$

$$=\frac{600}{x}-\frac{x}{4}$$

$$v = x^2h\ = x^{\cancel{2}}\left[\frac{2400-x^2}{4\cancel{x}}\right]$$

$$= 600\ x-\frac{1}{4}x^3$$

$$\frac{dv}{dx}=600-\frac{3}{4}x^2$$

$$\frac{dv}{dx}=0\Rightarrow 600-\frac{3}{4}x^2=0$$

$$x^2=\frac{2400}{3}=800$$

$$x = \sqrt{800}$$

$$\frac{d^2v}{dx^2}=\frac{-3}{2}x$$

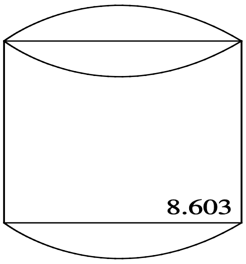
When $x=\sqrt{800}$, $\frac{d^2v}{dx^2}=-\text{ve}\ \Rightarrow$ maximum

Maximum volume = $600\ (\sqrt{800}\ )-\frac{1}{4}\left(\sqrt{800}\right)^3 = 11313.7\ \text{cm}^3$

**19** $V = 2000\ \text{cm}^3$

$$\pi r^2h = 2000$$

$$h=\frac{2000}{\pi r^2}$$



$$A = \pi r^2 + 2\pi rh$$

$= \pi r^2 + 2\pi r\left(\frac{2000}{\pi r^2}\right)$

$= \pi r^2 + \frac{4000}{r}$

$\frac{dA}{dr} = 2\pi r - \frac{4000}{r^2}$

$\frac{dA}{dr} = 0 \quad \Rightarrow 2\pi r = \frac{4000}{r^2}$

$r^3 = \frac{4000}{2\pi} \Rightarrow r = \sqrt[3]{\frac{4000}{2\pi}} = 8.603$

$h = \frac{2000}{\pi(8.603)^2}$

$= 8.602$