Chapter 13 Differentiation I

Try these 13.1

(a) $f(x) = \frac{1}{x}$

$$f(x+h) = \frac{1}{x+h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x-(x+h)}{(x+h)(x)}}{h}$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \times \frac{-\cancel{h}}{(x+h)(x)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= -\frac{1}{x(x)}$$

$$= -\frac{1}{x^2}$$

(b) $f(x) = \frac{1}{x^2}$

$$f(x+h) = \frac{1}{(x+h)^2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{-2x - h^2}{x^2(x+h)^2}$$

$$= \frac{-2x}{x^2(x+0)^2} = \frac{-2x}{x^4}$$

$$= \frac{-2}{x^3}$$

Try these 13.2

(a) $f(x) = x^{10}$

$f'(x) = 10x^9$

(b) $f(x) = x^{\frac{3}{4}}$

$f'(x) = \frac{3}{4}x^{-\frac{1}{4}}$

(c) $f(x) = \frac{1}{x^4} = x^{-4}$

$f'(x) = -4x^{-5} = \frac{-4}{x^5}$

(d) $f(x) = \frac{1}{x^8} = x^{-8}$

$f'(x) = -8x^{-9} = \frac{-8}{x^9}$

Try these 13.3

(a) $f(x) = 10x^5$

$f'(x) = 50x^4$

(b) $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}.$

(c) $f(x) = \frac{12}{x^3} = 12x^{-3}$

$f'(x) = -36x^{-4}$

$= \frac{-36}{x^4}$

Try these 13.4

$f(x) = g(x) - m(x)$

$$f'(x) = \lim_{h\to 0}\frac{[g(x+h) - m(x+h)] - [g(x) - m(x)]}{h}$$

$$= \lim_{h\to 0}\frac{[g(x+h) - g(x)] - [m(x+h) - m(x)]}{h}$$

$$= \lim_{h\to 0}\frac{[g(x+h) - g(x)]}{h} - \lim_{h\to 0}\frac{[m(x+h) - m(x)]}{h}$$

$= g'(x) - m'(x)$

Try these 13.5

(a) $y = \dfrac{x^3 - x^2}{x^5}$

$y = \dfrac{1}{x^2} - \dfrac{1}{x^3} = x^{-2} - x^{-3}$

$\dfrac{dy}{dx} = -2x^{-3} + 3x^{-4}$

$= \dfrac{-2}{x^3} + \dfrac{3}{x^4}$

(b) $y = \dfrac{4x + 1}{\sqrt{x}}$

$= \dfrac{4x}{\sqrt{x}} + \dfrac{1}{\sqrt{x}} = 4x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = 4\left(\dfrac{1}{2}\right) x^{\frac{1}{2}-1} + \left(-\dfrac{1}{2}\right) x^{-\frac{1}{2}-1}$

$= 2x^{-\frac{1}{2}} - \dfrac{1}{2}x^{-\frac{3}{2}}$

$= \dfrac{2}{\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}$

(c) $y = (x - 2)(3x + 4)$

$y = 3x^2 - 2x - 8$

$\dfrac{dy}{dx} = 6x - 2$

(d) $y = (x + 5)(5x - 1)$

$y = 5x^2 + 24x - 5$

$\dfrac{dy}{dx} = 10x + 24$

Try these 13.6

(a) $f(x) = 2x - 4$

$f(x + h) = 2(x + h) - 4$

$= 2x + 2h - 4$

$f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{2x + 2h - 4 - (2x - 4)}{h}$

$= \lim_{h \to 0} \left(\dfrac{2h}{h}\right)$

$= \lim_{h \to 0} 2 = 2$

$\therefore f'(x) = 2$

(b) $f(x) = 6x^2 + 2x + 1$

$f(x + h) = 6(x + h)^2 + 2(x + h) + 1$

$= 6(x^2 + 2xh + h^2) + 2x + 2h + 1$

$= 6x^2 + 12xh + 6h^2 + 2x + 2h + 1$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{6x^2 + 12xh + 6h^2 + 2x + 2h + 1 - (6x^2 + 2x + 1)}{h}$

$= \lim_{h \to 0} \frac{12xh + 6h^2 + 2h}{h}$

$= \lim_{h \to 0} (12x + 6h + 2)$

$= 12x + 2$

$\therefore f'(x) = 12x + 2$

(c) $f(x+h) = \frac{1}{2(x+h)+1}$, $f(x) = \frac{1}{2x+1}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\frac{1}{2x+2h+1} - \frac{1}{2x+1}}{h}$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2x + 1 - 2x - 2h - 1}{(2x+2h+1)(2x+1)}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-2h}{(2x+2h+1)(2x+1)}\right]$

$= \lim_{h \to 0} \frac{-2}{(2x+2h+1)(2x+1)}$

$= \frac{-2}{(2x+1)(2x+1)}$

$= \frac{-2}{(2x+1)^2}$

$\therefore f'(x) = \frac{-2}{(2x+1)^2}$

Exercise 13A

1 (a) $f(x) = 4x - 7$

$f(x+h) = 4(x+h) - 7 = 4x + 4h - 7$

By definition

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(4x + 4h - 7) - (4x - 7)}{h}$

$= \lim_{h \to 0} \left(\frac{4h}{h}\right)$

$= \lim_{h \to 0} 4 = 4$

(b) $f(x) = 3x + 9$

$f(x + h) = 3(x + h) + 9 = 3x + 3h + 9$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h + 9 - (3x + 9)}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h}$$

$$= \lim_{h \to 0} 3 = 3$$

(c) $f(x) = x^2 + 2x + 5$

$f(x + h) = (x + h)^2 + 2(x + h) + 5$

$= x^2 + 2xh + h^2 + 2x + 2h + 5$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h + \cancel{5}) - (\cancel{x^2} + \cancel{2x} + \cancel{5})}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h}$$

$$= \lim_{h \to 0} (2x + h + 2)$$

$= 2x + 2 + 0$

$= 2x + 2$

(d) $f(x) = 3x^2 - 4x + 1$

$f(x + h) = 3(x + h)^2 - 4(x + h) + 1$

$= 3[x^2 + 2xh + h^2] - 4x - 4h + 1$

$= 3x^2 + 6xh + 3h^2 - 4x - 4h + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{4x} - 4h + \cancel{1} - (\cancel{3x^2} - \cancel{4x} + \cancel{1})}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 4h}{h}$$

$$= \lim_{h \to 0} (6x + 3h - 4)$$

$= 6x + 0 - 4$

$= 6x - 4$

(e) $f(x) = 5x^2 + 2$

$f(x + h) = 5(x + h)^2 + 2$

$= 5x^2 + 10xh + 5h^2 + 2$

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0} \frac{\cancel{5x^2} + 10xh + 5h^2 + \cancel{2} - \cancel{5x^2} - \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2}{h}$$

$$= \lim_{h \to 0} (10x + 5h)$$

$= 10x + 5(0)$
$= 10x$

(f) $f(x) = x^3 + 3x + 1$

$f(x+h) = (x+h)^3 + 3(x+h) + 1$

$= x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{3x} + 3h + \cancel{1} - \cancel{x^3} - \cancel{3x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 3h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 + 3)$$

$= 3x^2 + 3x(0) + (0)^2 + 3$

$= 3x^2 + 3$

(g) $f(x) = \dfrac{1}{(x+2)^3}$

$f(x+h) = \dfrac{1}{(x+h+2)^3}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{(x+h+2)^3} - \dfrac{1}{(x+2)^3}}{h}$$

$$= \lim_{h \to 0} \frac{(x+2)^3 - (x+2+h)^3}{h(x+2)^3(x+2+h)^3}$$

$$= \lim_{h \to 0} \frac{(x+2)^3 - (x+2)^3 - 3(x+2)^2h - 3(x+2)h^2 - h^3}{h(x+2)^3(x+2+h)^3}$$

$$= \lim_{h \to 0} \frac{\cancel{h}[-3(x+2)^2 - 3(x+2)h - h^2]}{\cancel{h}(x+2)^3(x+2+h)^3}$$

$$= \lim_{h \to 0} \frac{-3(x+2)^2 - 3(x+2)h - h^2}{(x+2)^3(x+2+h)^3}$$

$$= \frac{-3(x+2)^2 - 3(x+2)(0) - 0^2}{(x+2)^3(x+2+0)^3} = \frac{-3(x+2)^2}{(x+2)^6} = \frac{-3}{(x+2)^4}$$

2 (a) $\dfrac{d}{dx}[4x^3 + 5x - 6] = 12x^2 + 5$

(b) $\dfrac{d}{dx}[x^5 - 3x^2 + 2] = 5x^4 - 6x$

(c) $\dfrac{d}{dx}[x^5 + 7x^3 + 2x + 4] = 5x^4 + 21x^2 + 2$

(d) $\dfrac{d}{dx}\left[4x + \dfrac{3}{x}\right] = \dfrac{d}{dx}[4x + 3x^{-1}] = 4 - \dfrac{3}{x^2}$

(e) $\dfrac{d}{dx}\left[6x^2 - \dfrac{7}{x^3}\right] = \dfrac{d}{dx}[6x^2 - 7x^{-3}] = 12x + \dfrac{21}{x^4}$

(f) $\frac{d}{dx}\left[\frac{4}{x^3}+\frac{3}{x^2}-\frac{1}{x}\right]=\frac{d}{dx}[4x^{-3}+3x^{-2}-x^{-1}]=-12x^{-4}-6x^{-3}+x^{-2}=\frac{-12}{x^4}-\frac{6}{x^3}+\frac{1}{x^2}$

(g) $\frac{d}{dx}\left[4x+3\sqrt{x}-2\right]=\frac{d}{dx}\left[4x+3x^{\frac{1}{2}}-2\right]=4+\frac{3}{2}x^{-\frac{1}{2}}=4+\frac{3}{2\sqrt{x}}$

(h) $\frac{d}{dx}\left[6x+\frac{1}{\sqrt{x}}-4\right]=\frac{d}{dx}\left[6x+x^{-\frac{1}{2}}-4\right]=6-\frac{1}{2}x^{-\frac{3}{2}}=6-\frac{1}{2(\sqrt{x})^3}$

(i) $\frac{d}{dx}\left[2x^{\frac{5}{2}}-3x^{\frac{3}{2}}+x^{\frac{1}{2}}\right]=5x^{\frac{3}{2}}-\frac{9}{2}x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$

(j) $\frac{d}{dx}\left[6x\sqrt{x}+5\sqrt{x}\right]=\frac{d}{dx}\left[6x^{\frac{3}{2}}+5x^{\frac{1}{2}}\right]=9x^{\frac{1}{2}}+\frac{5}{2}x^{-\frac{1}{2}}=9\sqrt{x}+\frac{5}{2\sqrt{x}}$

(k) $\frac{d}{dx}\left[6x^2\sqrt{x}-\frac{3}{\sqrt{x}}\right]=\frac{d}{dx}\left[6x^{\frac{5}{2}}-3x^{-\frac{1}{2}}\right]=15x^{\frac{3}{2}}+\frac{3}{2}x^{-\frac{3}{2}}$

(l) $\frac{d}{dx}\left[4+\frac{3}{x}\right]=\frac{d}{dx}\left[4+3x^{-1}\right]=\frac{-3}{x^2}$

3 (a) $y=\frac{x^2+3x}{x}$

$=x+3$

$\frac{dy}{dx}=1$

(b) $y=\frac{4x^3-3x^2+2}{x^4}$

$=\frac{4x^3}{x^4}-\frac{3x^2}{x^4}+\frac{2}{x^4}$

$=4x^{-1}-3x^{-2}+2x^{-4}$

$\frac{dy}{dx}=-4x^{-2}+6x^{-3}-8x^{-5}$

$=\frac{-4}{x^2}+\frac{6}{x^3}-\frac{8}{x^5}$

(c) $y=\frac{x^3+6x}{x^2}$

$=x+\frac{6}{x}$

$\frac{dy}{dx}=1-\frac{6}{x^2}$

(d) $y=\frac{6x+1}{\sqrt{x}}$

$=6x^{\frac{1}{2}}+x^{-\frac{1}{2}}$

$\frac{dy}{dx}=3x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$

(e) $y=\frac{3x\sqrt{x}+2}{\sqrt{x}}$

$=3x+2x^{-\frac{1}{2}}$

$\frac{dy}{dx}=3-x^{-\frac{3}{2}}$

(f) $y = \dfrac{7x^2 - 4x + 5}{2x}$

$= \dfrac{7}{2}x - 2 + \dfrac{5}{2}x^{-1}$

$\dfrac{dy}{dx} = \dfrac{7}{2} - \dfrac{5}{2}x^{-2}$

$= \dfrac{7}{2} - \dfrac{5}{2x^2}$

4 (a) $y = (x + 2)(7x - 1)$

$= 7x^2 + 13x - 2$

$\dfrac{dy}{dx} = 14x + 13$

(b) $y = (3x + 4)(2x + 7)$

$= 6x^2 + 29x + 28$

$\dfrac{dy}{dx} = 12x + 29$

(c) $y = \left(\sqrt{x} - 1\right)(x + 2)$

$= x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - x - 2$

$\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 1$

(d) $y = (6x + 2)\left(\sqrt{x} + 3\right)$

$= 6x^{\frac{3}{2}} + 18x + 2x^{\frac{1}{2}} + 6$

$\dfrac{dy}{dx} = 9x^{\frac{1}{2}} + 18 + x^{-\frac{1}{2}}$

$= 9\sqrt{x} + 18 + \dfrac{1}{\sqrt{x}}$

(e) $y = \dfrac{(4x - 1)(x + 3)}{x}$

$= \dfrac{4x^2 + 12x - x - 3}{x}$

$= 4x + 11 - 3x^{-1}$

$\dfrac{dy}{dx} = 4 + 3x^{-2} = 4 + \dfrac{3}{x^2}$

(f) $y = \dfrac{(4x - 2)x^{\frac{1}{2}}}{x}$

$= 4x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = 2x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

(g) $y = \left(3\sqrt{x} + 4x^{\frac{3}{2}}\right)x$

$= 3x^{\frac{3}{2}} + 4x^{\frac{5}{2}}$

$\dfrac{dy}{dx} = \dfrac{9}{2}x^{\frac{1}{2}} + 10x^{\frac{3}{2}}$

(h) $y = \dfrac{3x-5}{\sqrt{x}}$

$y = 3x^{\frac{1}{2}} - 5x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{3}{2}x^{-\frac{1}{2}} + \dfrac{5}{2}x^{-\frac{3}{2}}$

5 (a) $y = 3x^3 - 4x^2 + 2$

$\dfrac{dy}{dx} = 9x^2 - 8x$

$x = 1,\ \dfrac{dy}{dx} = 9 - 8 = 1$

(b) $y = 4x + 3x^2$

$\dfrac{dy}{dx} = 4 + 6x$

$x = 1,\ \dfrac{dy}{dx} = 10$

(c) $y = (2x + 1)(x - 2)$

$= 2x^2 - 3x - 2$

$\dfrac{dy}{dx} = 4x - 3$

$x = 2,\ \dfrac{dy}{dx} = 8 - 3 = 5$

(d) $y = \sqrt{x}\,(x + 4)$

$= x^{\frac{3}{2}} + 4x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$

$x = 16,\ \dfrac{dy}{dx} = \dfrac{3}{2}(4) + \dfrac{2}{4} = 6\dfrac{1}{2}$

6 $y = 4x^2 + x - 2$

$\dfrac{dy}{dx} = 8x + 1$

$\dfrac{dy}{dx} = 17$

$8x + 1 = 17$

$x = 2$

$y = 4(2)^2 + 2 - 2 = 16$

(2, 16)

7 $y = 4x^3 - 3x^2 + 2x + 1$

$\dfrac{dy}{dx} = 12x^2 - 6x + 2$

$\dfrac{dy}{dx} = 8 \quad \Rightarrow 12x^2 - 6x + 2 = 8$

$12x^2 - 6x - 6 = 0$

$2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$

$x = -\dfrac{1}{2}, 1$

$x = -\frac{1}{2},\ y = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) + 1$

$= -\frac{1}{2} - \frac{3}{4} - 1 + 1$

$= -1\frac{1}{4}$

$x = 1,\ y = 4 - 3 + 2 + 1 = 4$

$\left(-\frac{1}{2}, -\frac{5}{4}\right)$ and $(1, 4)$

8 $y = ax^3 + bx$

$\frac{dy}{dx} = 3ax^2 + b$

When $x = 1,\ y = 2$

$\Rightarrow 2 = a + b$ [1]

$x = 1,\ \frac{dy}{dx} = 4 \Rightarrow 3a + b = 4$ [2]

[2] – [1] $\Rightarrow 2a = 2,\ a = 1,\ b = 1$

9 $y = \frac{a}{x^2} + bx$

$x = 2,\ y = -13 \Rightarrow \frac{a}{4} + 2b = -13$ [1]

$\frac{dy}{dx} = -2ax^{-3} + b$

$x = 2,\ \frac{dy}{dx} = -8 \Rightarrow -8 = -\frac{1}{4}a + b$ [2]

[1] – [2] $\Rightarrow 3b = -21,\quad b = -7$

$-8 = -\frac{1}{4}a - 7$

$-1 = -\frac{1}{4}a$

$a = 4$

$a = 4,\ b = -7$

10 $y = \frac{p}{x} + qx$

$x = 3,\ y = 13 \Rightarrow 13 = \frac{1}{3}p + 3q$ [1]

Gradient of the line $= \frac{11}{3}$

$\frac{dy}{dx} = \frac{11}{3}$

$\frac{dy}{dx} = -px^{-2} + q$

$\frac{11}{3} = -\frac{1}{9}p + q$

$11 = -\frac{1}{3}p + 3q$ [2]

[1] + [2] $\Rightarrow 24 = 6q \Rightarrow q = 4$

$11 = -\frac{1}{3}p + 12$

$p = 3$

$p = 3, q = 4$

11 (a) $y = 4x + \frac{1}{x}$

$\frac{dy}{dx} = 4 - \frac{1}{x^2}$

$x = 1, \frac{dy}{dx} = 4 - 1 = 3$

(b) $\frac{dy}{dx} = 0 \Rightarrow 4 - \frac{1}{x^2} = 0$

$\frac{1}{x^2} = 4$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$x = \frac{1}{2},\ y = 4\left(\frac{1}{2}\right) + \frac{1}{½} = 2 + 2 = 4$

$x = -\frac{1}{2},\ \ y = 4\left(-\frac{1}{2}\right) + \frac{1}{-½} = -2 - 2 = -4$

$\left(\frac{1}{2}, 4\right)\left(-\frac{1}{2}, -4\right)$

Try these 13.7

(a) (i) $y = (6x + 2)^7$

$\frac{dy}{dx} = 7(6)\,(6x + 2)^6 = 42\,(6x + 2)^6$

(ii) $y = (3 - 2x)^{\frac{3}{4}}$

$\frac{dy}{dx} = \frac{3}{4}(-2)\,(3 - 2x)^{-\frac{1}{4}} = -\frac{3}{2}(3 - 2x)^{-\frac{1}{4}}$

(b) $y = 4(3x - 2)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}(4)(3)\,(3x - 2)^{-\frac{3}{2}}$

$= -6(3x - 2)^{-\frac{3}{2}}$

when $x = 6, \frac{dy}{dx} = -6\,(16)^{-\frac{3}{2}} = \frac{-6}{64} = \frac{-3}{32}$

Try these 13.8

(a) $y = x^2\,(3x + 2)^5$

$u = x^2, v = (3x + 2)^5$

$\frac{du}{dx} = 2x, \frac{dv}{dx} = 15(3x + 2)^4$

$\frac{dy}{dx} = x^2[15(3x + 2)^4] + (3x + 2)^5 (2x)$

$= x(3x + 2)^4 [15x + 2(3x + 2)]$

$= x(3x + 2)^4 (21x + 4)$

(b) $y = (4x + 1)(6x - 2)^{\frac{1}{2}}$

$u = 4x + 1, \; v = (6x - 2)^{\frac{1}{2}}$

$\frac{du}{dx} = 4, \quad \frac{dv}{dx} = \frac{1}{2}(6x - 2)^{-\frac{1}{2}}(6)$

$= \frac{3}{\sqrt{6x - 2}}$

$\frac{dy}{dx} = \frac{3(4x + 1)}{\sqrt{6x - 2}} + 4\sqrt{6x - 2}$

$= \frac{3(4x + 1) + 4(6x - 2)}{\sqrt{6x - 2}}$

$= \frac{36x - 5}{\sqrt{6x - 2}}$

(c) $y = (4x^2 + 6x + 1)^5 (6x + 2)$

$u = (4x^2 + 6x + 1)^5, v = 6x + 2$

$\frac{du}{dx} = 5(8x + 6)(4x^2 + 6x + 1)^4, \frac{dv}{dx} = 6$

$\frac{dy}{dx} = 6(4x^2 + 6x + 1)^5 + (6x + 2)[5(8x + 6)](4x^2 + 6x + 1)^4$

$= (4x^2 + 6x + 1)^4 [6(4x^2 + 6x + 1) + 5(6x + 2)(8x + 6)]$

$= (4x^2 + 6x + 1)^4 [24x^2 + 36x + 6 + 240x^2 + 260x + 60]$

$= (4x^2 + 6x + 1)^4 (264x^2 + 296x + 66)$

Try these 13.9

(a) (i) $y = \frac{6x + 2}{x^2 + 1}$

$u = 6x + 2, v = x^2 + 1$

$\frac{du}{dx} = 6, \frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{(x^2 + 1)(6) - (6x + 2)(2x)}{(x^2 + 1)^2}$

$= \frac{6x^2 + 6 - 12x^2 - 4x}{(x^2 + 1)^2}$

$= \frac{-6x^2 - 4x + 6}{(x^2 + 1)^2}$

(ii) $y = \frac{3x^2 - 2}{x^3 + 4}$

$u = 3x^2 - 2,\ v = x^3 + 4$

$\frac{du}{dx} = 6x,\quad \frac{dv}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{(x^3 + 4)(6x) - (3x^2 - 2)(3x^2)}{(x^3 + 4)^2}$

$= \frac{6x^4 + 24x - 9x^4 + 6x^2}{(x^3 + 4)^2}$

$= \frac{-3x^4 + 6x^2 + 24x}{(x^3 + 4)^2}$

(b) $y = \frac{x + 1}{(x - 2)^{\frac{1}{2}}}$

$u = x + 1,\ v = (x - 2)^{\frac{1}{2}}$

$\frac{du}{dx} = 1, \frac{dv}{dx} = \frac{1}{2}(x - 2)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{(x - 2)^{\frac{1}{2}} - (x + 1)\frac{1}{2}(x - 2)^{-\frac{1}{2}}}{[(x - 2)^{\frac{1}{2}}]^2}$

$x = 11, \frac{dy}{dx} = \frac{9^{\frac{1}{2}} - \frac{1}{2}(12)(9)^{-\frac{1}{2}}}{9} = \frac{3 - 2}{9} = \frac{1}{9}$

Try these 13.10

$y = \cos x$

$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}$

$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

$= \lim_{h \to 0} \frac{\cos(\cos h - 1) - \sin x \sin h}{h}$

$= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}$

$= \cos x\ (0) - \sin x\ (1)$

$= -\sin x$

Try these 13.11

(a) $y = \cot x = \frac{\cos x}{\sin x}$

$u = \cos x,\ v = \sin x$

$\frac{du}{dx} = -\sin x, \frac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$
$$= \frac{-[\sin^2 x + \cos^2 x]}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

(b) $y = \operatorname{cosec} x = (\sin x)^{-1}$

$$\frac{dy}{dx} = -(\sin x)^{-2}(\cos x)$$
$$= -\frac{\cos x}{\sin^2 x}$$
$$= -\frac{\cos x}{\sin x} \times \frac{1}{\sin x}$$
$$= -\operatorname{cosec} x \cot x$$

Try these 13.12

(a) $y = \sin 4x$

$$\frac{dy}{dx} = 4\cos 4x$$

(b) $y = \cos(4x + \pi)$

$$\frac{dy}{dx} = -4\sin(4x + \pi)$$

(c) $y = x^2 \cos\left(3x - \frac{\pi}{2}\right)$

$$u = x^2,\ v = \cos\left(3x - \frac{\pi}{2}\right)$$
$$\frac{du}{dx} = 2x,\ \frac{dv}{dx} = -3\sin\left(3x - \frac{\pi}{2}\right)$$
$$\frac{dy}{dx} = -3x^2 \sin\left(3x - \frac{\pi}{2}\right) + 2x\cos\left(3x - \frac{\pi}{2}\right)$$

(d) $y = x \cot x$

$$u = x,\ \frac{dv}{dx} = \cot x$$
$$\frac{du}{dx} = 1,\ v = -\operatorname{cosec}^2 x$$
$$\frac{dy}{dx} = -x\operatorname{cosec}^2 x + \cot x$$

(e) $y = x^2 \sec x$

$u = x^2$, $v = \sec x$

$$\frac{du}{dx} = 2x,\ \frac{dv}{dx} = \sec x \tan x$$
$$\frac{dy}{dx} = x^2 \sec x \tan x + 2x \sec x$$

Exercise 13B

1 (a) $y = (6x + 1)^5$

$\frac{dy}{dx} = 30(6x + 1)^4$

(b) $y = (4 - 3x)^6$

$\frac{dy}{dx} = -18(4 - 3x)^5$

(c) $y = (2x + 9)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(2)(2x + 9)^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2x + 9}}$

(d) $y = \frac{4}{\sqrt{5x + 1}} = 4(5x + 1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = 4\left(-\frac{1}{2}\right)(5)(5x + 1)^{-\frac{3}{2}}$

$= -10(5x + 1)^{-\frac{3}{2}}$

(e) $y = (6x^2 + 3x + 1)^4$

$\frac{dy}{dx} = 4(12x + 3)(6x^2 + 3x + 1)^3$

(f) $y = (6x^3 + 5x)^{\frac{1}{4}}$

$\frac{dy}{dx} = \frac{1}{4}(18x^2 + 5)(6x^3 + 5x)^{-\frac{3}{4}}$

(g) $y = 3(7x^2 - 5)^{-\frac{1}{2}}$

$\frac{dy}{dx} = 3\left(-\frac{1}{2}\right)(14x)(7x^2 - 5)^{-\frac{3}{2}}$

$= -21x(7x^2 - 5)^{-\frac{3}{2}}$

(h) $y = (4x^{\frac{1}{2}} + 5)^{10}$

$\frac{dy}{dx} = 10(2x^{-\frac{1}{2}})(4x^{\frac{1}{2}} + 5)^9$

$= \frac{20(4\sqrt{x} + 5)^9}{\sqrt{x}}$

(i) $y = (7x^{\frac{1}{2}} + 3x^5)^7$

$\frac{dy}{dx} = 7\left(\frac{7}{2}x^{-\frac{1}{2}} + 15x^4\right)(7x^{\frac{1}{2}} + 3x^5)^6$

(j) $y = (x^3 - 5x + 2)^{\frac{3}{4}}$

$\frac{dy}{dx} = \frac{3}{4}(3x^2 - 5)(x^3 - 5x + 2)^{-\frac{1}{4}}$

2 (a) $\theta = (3t + 1)(4t + 2)^8$

$u = 3t + 1, \ v = (4t + 2)^8$

$\frac{du}{dt} = 3, \ \frac{dv}{dt} = 32(4t + 2)^7$

$$\frac{d\theta}{dt} = 32\,(3t+1)\,(4t+2)^7 + 3\,(4t+2)^8$$
$$= (4t+2)^7[32\,(3t+1) + 3(4t+2)]$$
$$= (4t+2)^7[96t + 32 + 12t + 6]$$
$$= (4t+2)^7\,(108t + 38)$$

(b) $\theta = t^2(7t+1)^3$

$u = t^2,\ v = (7t+1)^3$

$$\frac{du}{dt} = 2t,\ \frac{dv}{dt} = 21(7t+1)^2$$
$$\frac{d\theta}{dt} = 21t^2(7t+1)^2 + 2t(7t+1)^3$$
$$= t(7t+1)^2[21t + 2(7t+1)]$$
$$= t(7t+1)^2\,(35t+2)$$

(c) $\theta = 4t\sqrt{2t-1} = 4t(2t-1)^{\frac{1}{2}}$

$$\frac{d\theta}{dt} = 4(2t-1)^{\frac{1}{2}} + (4t)\left(\frac{1}{2}\right)(2)\,(2t-1)^{-\frac{1}{2}}$$
$$= 4\sqrt{2t-1} + \frac{4t}{\sqrt{2t-1}}$$
$$= \frac{4(2t-1)+4t}{\sqrt{2t-1}}$$
$$= \frac{12t-4}{\sqrt{2t-1}}$$

(d) $\theta = 6t^2\,(t^3+2t)^{\frac{1}{2}}$

$$\frac{d\theta}{dt} = 12t\left(t^3+2t\right)^{\frac{1}{2}} + \left(6t^2\right)\left(\frac{1}{2}\right)\left(3t^2+2\right)\left(t^3+2t\right)^{-\frac{1}{2}}$$
$$= 12t\sqrt{t^3+2t} + \frac{3t^2(3t^2+2)}{\sqrt{t^3+2t}}$$
$$= \frac{12t(t^3+2t) + 3t^2(3t^2+2)}{\sqrt{t^3+2t}}$$
$$= \frac{21t^4+30t^2}{\sqrt{t^3+2t}} = \frac{3t^2(7t^2+10)}{\sqrt{t^3+2t}}$$

(e) $\theta = (5t^2+2)(7-3t)^4$

$$\frac{d\theta}{dt} = 10t\,(7-3t)^4 + (-12)\,(5t^2+2)(7-3t)^3$$
$$= (7-3t)^3[10t\,(7-3t) - 12(5t^2+2)]$$
$$= (7-3t)^3[-90t^2 + 70t - 24]$$

(f) $\theta = t^3(4t^3+3t^2+1)^{\frac{1}{4}}$

$$\frac{d\theta}{dt} = 3t^2(4t^3+3t^2+1)^{\frac{1}{4}} + (t^3)\left(\frac{1}{4}\right)(12t^2+6t)(4t^3+3t^2+1)^{-\frac{3}{4}}$$
$$= 3t^2(4t^3+3t^2+1)^{\frac{1}{4}} + \frac{1}{4}t^4(12t+6)(4t^3+3t^2+1)^{-\frac{3}{4}}$$

(g) $\theta = (7t^{\frac{1}{2}} + 3t)(3t^2 + 1)^{-\frac{1}{2}}$

$\frac{d\theta}{dt} = \left(\frac{7}{2}t^{-\frac{1}{2}} + 3\right)(3t^2 + 1)^{-\frac{1}{2}} + (7t^{\frac{1}{2}} + 3t)\left(-\frac{1}{2}\right)(6t)(3t^2 + 1)^{-\frac{3}{2}}$

$= \left(\frac{7}{2}t^{-\frac{1}{2}} + 3\right)(3t^2 + 1)^{-\frac{1}{2}} - 3t(7t^{\frac{1}{2}} + 3t)(3t^2 + 1)^{-\frac{3}{2}}$

(h) $\theta = t^{\frac{1}{2}}(t + 5)^{\frac{3}{4}}$

$\frac{d\theta}{dt} = \frac{1}{2}t^{-\frac{1}{2}}(t + 5)^{\frac{3}{4}} + \frac{3}{4}t^{\frac{1}{2}}(t + 5)^{-\frac{1}{4}}$

$= \frac{1}{2\sqrt{t}}(t + 5)^{\frac{3}{4}} + \frac{3}{4}\sqrt{t}\,(t + 5)^{-\frac{1}{4}}$

3 (a) $y = \frac{4x + 2}{x - 1}$

$\frac{dy}{dx} = \frac{(x - 1)(4) - (4x + 2)}{(x - 1)^2}$

$= \frac{4x - 4 - 4x - 2}{(x - 1)^2}$

$= \frac{-6}{(x - 1)^2}$

(b) $y = \frac{3x - 5}{6x + 2}$

$\frac{dy}{dx} = \frac{(6x + 2)(3) - (3x - 5)(6)}{(6x + 2)^2}$

$= \frac{18x + 6 - 18x + 30}{(6x + 2)^2}$

$= \frac{36}{(6x + 2)^2}$

(c) $y = \frac{x^2 + 1}{2x + 5}$

$\frac{dy}{dx} = \frac{(2x + 5)(2x) - (x^2 + 1)(2)}{(2x + 5)^2}$

$= \frac{4x^2 + 10x - 2x^2 - 2}{(2x + 5)^2}$

$= \frac{2x^2 + 10x - 2}{(2x + 5)^2}$

(d) $y = \frac{3x + 2}{\sqrt{4x - 1}}$

$\frac{dy}{dx} = \frac{(\sqrt{4x} - 1)(3) - (3x + 2)\left(\frac{1}{2}\right)(4)(4x - 1)^{-\frac{1}{2}}}{(\sqrt{4x - 1})^2}$

$= \frac{3\sqrt{4x - 1} - \frac{2(3x + 2)}{\sqrt{4x - 1}}}{4x - 1}$

$$= \frac{3(4x-1)-6x-4}{(4x-1)\sqrt{4x-1}}$$

$$= \frac{6x-7}{(4x-1)\sqrt{4x-1}}$$

(e) $y = \dfrac{x^3-2x+1}{4x^2+5}$

$$\frac{dy}{dx} = \frac{(4x^2+5)(6x^2-2)-(x^3-2x+1)(8x)}{(4x^2+5)^2}$$

$$= \frac{24x^4-8x^2+30x^2-10-8x^4+16x^2-8x}{(4x^2+5)^2}$$

$$= \frac{16x^4+38x^2-8x-10}{(4x^2+5)^2}$$

(f) $y = \dfrac{x^4-2}{\sqrt{3x+1}}$

$$\frac{dy}{dx} = \frac{\sqrt{3x+1}\,(4x^3)-(x^4-2)\left(\frac{1}{2}\right)(3)(3x+1)^{-\frac{1}{2}}}{(\sqrt{3x+1})^2}$$

$$= \frac{4x^3\sqrt{3x+1}-\dfrac{\frac{3}{2}(x^4-2)}{\sqrt{3x+1}}}{3x+1} = \frac{4x^3(3x+1)-\frac{3}{2}x^4+3}{(3x+1)^{\frac{3}{2}}} = \frac{\frac{21}{2}x^4+4x^3+3}{(3x+1)^{\frac{3}{2}}}$$

(g) $y = \dfrac{x}{(x+3)^3}$

$$\frac{dy}{dx} = \frac{(x+3)^3-x(3)(x+3)^2}{(x+3)^6}$$

$$= \frac{(x+3)^2[x+3-3x]}{(x+3)^6}$$

$$= \frac{-2x+3}{(x+3)^4}$$

(h) $y = \dfrac{x^4+2x}{x^2-7}$

$$\frac{dy}{dx} = \frac{(x^2-7)(4x^3+2)-(x^4+2x)(2x)}{(x^2-7)^2}$$

$$= \frac{4x^5+2x^2-28x^3-14-2x^5-4x^2}{(x^2-7)^2}$$

$$= \frac{2x^5-28x^3-2x^2-14}{(x^2-7)^2}$$

(i) $y = \dfrac{(3x+2)^2}{(2x-1)^3}$

$$\frac{dy}{dx} = \frac{(2x-1)^3(6)(3x+2)-(3x+2)^2(6)(2x-1)^2}{(2x-1)^6}$$

$= \frac{6(2x-1)^2(3x+2)[2x-1-(3x+2)]}{(2x-1)^6}$

$= \frac{6(3x+2)(-x-3)}{(2x-1)^4}$

(j) $y = \frac{7x+3}{5x-1}$

$\frac{dy}{dx} = \frac{(5x-1)(7)-(7x+3)(5)}{(5x-1)^2}$

$= \frac{35x-7-35x-15}{(5x-1)^2} = \frac{-22}{(5x-1)^2}$

4 (a) $y = (2x-1)^5$

$\frac{dy}{dx} = 10(2x-1)^4$

$x = 1, \frac{dy}{dx} = 10$

(b) $y = x^2(x+1)^3$

$\frac{dy}{dx} = 2x(x+1)^3 + 3x^2(x+1)^2$

$x = 0, \frac{dy}{dx} = 0$

(c) $y = \frac{x^2+1}{x+2}$

$\frac{dy}{dx} = \frac{(x+2)(2x)-(x^2+1)(1)}{(x+2)^2}$

$= \frac{2x^2+4x-x^2-1}{(x+2)^2}$

$= \frac{x^2+4x-1}{(x+2)^2}$

$x = 2, \frac{dy}{dx} = \frac{4+8-1}{4^2} = \frac{11}{16}$

(d) $y = (x^2-5x+2)^4$

$\frac{dy}{dx} = 4(2x-5)(x^2-5x+2)^3$

$x = 0, \frac{dy}{dx} = (4)(-5)(2)^3 = -160$

5 (a) $\frac{d}{dx}[\sin 4x] = 4\cos 4x$

(b) $\frac{d}{dx}[\sin 6x] = 6\cos 6x$

(c) $\frac{d}{dx}[\cos 3x] = -3\sin 3x$

(d) $\frac{d}{dx}[\cos 7x] = -7\sin 7x$

(e) $\frac{d}{dx}[\cos 9x] = -9\sin 9x$

(f) $\frac{d}{dx}\left[\sin\left(3x+\frac{\pi}{4}\right)\right] = 3\cos\left(3x+\frac{\pi}{4}\right)$

(g) $\frac{d}{dx}\sin\left(\frac{\pi}{2}-4x\right) = -4\cos\left(\frac{\pi}{2}-4x\right)$

(h) $\frac{d}{dx}\sin\left(\frac{3\pi}{2}+4x\right) = 4\cos\left(\frac{3\pi}{2}+4x\right)$

(i) $\frac{d}{dx}\sin(8x+2) = 8\cos(8x+2)$

(j) $\frac{d}{dx}\cos(3x-\pi) = -3\sin(3x-\pi)$

(k) $\frac{d}{dx}\cos(5-7x) = 7\sin(5-7x)$

(l) $\frac{d}{dx}\cos(2\pi-9x) = 9\sin(2\pi-9x)$

6 (a) $\frac{d}{dx}\tan 2x = 2\sec^2 2x$

(b) $\frac{d}{dx}\tan 5x = 5\sec^2(5x)$

(c) $\frac{d}{dx}\tan(2x+\pi) = 2\sec^2(2x+\pi)$

(d) $\frac{d}{dx}\tan\left(3x-\frac{\pi}{2}\right) = 3\sec^2\left(3x-\frac{\pi}{2}\right)$

(e) $\frac{d}{dx}\sec(4x) = 4\sec(4x)\tan(4x)$

(f) $\frac{d}{dx}\sec(4x+3) = 4\sec(4x+3)\tan(4x+3)$

(g) $\frac{d}{dt}\cot(6x-\pi) = -6\,\text{cosec}^2(6x-\pi)$

(h) $\frac{d}{dx}\cot\left(\frac{3}{4}x-\frac{\pi}{4}\right) = -\frac{3}{4}\text{cosec}^2\left(\frac{3}{4}x-\frac{\pi}{4}\right)$

(i) $\frac{d}{dx}\text{cosec}\left(x-\frac{\pi}{4}\right) = -\text{cosec}\left(x-\frac{\pi}{4}\right)\cot\left(x-\frac{\pi}{4}\right)$

(j) $\frac{d}{dx}\text{cosec}(7x-4) = -7\,\text{cosec}(7x-4)\cot(7x-4)$

7 (a) $y = \sin x^2$

$\frac{dy}{dx} = 2x\cos x^2$

(b) $\theta = \sin(t^2+3)$

$\frac{d\theta}{dt} = 2t\cos(t^2+3)$

(c) $\theta = \cos(4x^2+\pi)$

$\frac{d\theta}{dx} = -8x\sin(4x^2+\pi)$

(d) $y = \tan(7x^3 - 8)$

$\frac{dy}{dx} = 21x^2 \sec^2(7x^3 - 8)$

(e) $v = 8\tan(3x^3 - 4x + 5)$

$\frac{dv}{dx} = 8(9x^2 - 4)\sec^2(3x^3 - 4x + 5)$

(f) $y = (7x + 5)^{10}$, $\frac{dy}{dx} = 70(7x + 5)^9$

(g) $y = (4x + 2)^{-3}$, $\frac{dy}{dx} = -12(4x + 2)^{-4} = \frac{-12}{(4x + 2)^4}$

(h) $y = 8(7x^2 + 5x + 1)^{-6}$

$\frac{dy}{dx} = -48(14x + 5)(7x^2 + 5x + 1)^{-7}$

(i) $x = \left(t^3 - \frac{1}{4}t^{-2}\right)^{\frac{1}{2}}$

$\frac{dx}{dt} = \frac{1}{2}\left(3t^2 + \frac{1}{2}t^{-3}\right)\left(t^3 - \frac{1}{4}t^{-2}\right)^{-\frac{1}{2}}$

$= \left(\frac{3}{2}t^2 + \frac{1}{4t^3}\right)\frac{1}{\sqrt{t^3 - \frac{1}{4t^2}}}$

(j) $y = \tan(5x + 1)^6$

$\frac{dy}{dx} = 30(5x + 1)^5 \sec^2(5x + 1)^6$

(k) $y = 5\cos\left(6x^2 + \frac{1}{x}\right)$

$\frac{dy}{dx} = -5\left(12x - \frac{1}{x^2}\right)\sin\left(6x^2 + \frac{1}{x}\right)$

(l) $y = \sec(x^3 + 5)$

$\frac{dy}{dx} = 3x^2 \sec(x^3 + 5)\tan(x^3 + 5)$

8 (a) $y = x \sin x$

$\frac{dy}{dx} = \sin x + x\cos x$

(b) $y = x \cos x$

$\frac{dy}{dx} = \cos x - x\sin x$

(c) $y = x^2 \tan x$

$\frac{dy}{dx} = 2x\tan x + x^2\sec^2 x$

(d) $y = x^3 \tan(3x + 2)$

$\frac{dy}{dx} = 3x^2\tan(3x + 2) + 3x^3\sec^2(3x + 2)$

(e) $y = (4x + 1)\sin 4x$

$\frac{dy}{dx} = 4\sin 4x + 4(4x + 1)\cos 4x$

(f) $y = x\tan(x^2)$

$$\frac{dy}{dx} = \tan(x^2) + 2x^2\sec^2(x^2)$$

(g) $y = \sin 2x \tan 2x$

$$\frac{dy}{dx} = 2\sin 2x \sec^2 2x + 2\cos 2x \tan 2x$$

$$= 2\sin 2x \sec^2 2x + 2\sin 2x = 2\sin 2x(\sec^2(2x) + 1)$$

(h) $y = (3x^2 + 1)\cos x$

$$\frac{dy}{dx} = 6x\cos x - (3x^2 + 1)\sin x$$

(i) $y = \left(\frac{1}{3}x^3 - 2x\right)\sec x$

$$\frac{dy}{dx} = (x^2 - 2)\sec x + \left(\frac{1}{3}x^3 - 2x\right)\sec x \tan x$$

(j) $y = \left(\frac{4}{x} + 2x\right)\cot 4x$

$$\frac{dy}{dx} = -4\left(\frac{4}{x} + 2x\right)\operatorname{cosec}^2(4x) + \left(\frac{-4}{x^2} + 2\right)\cot(4x)$$

(k) $y = \left(\frac{6}{x^2} - 3x + 2\right)\operatorname{cosec} x$

$$\frac{dy}{dx} = -\left(\frac{6}{x^2} - 3x + 2\right)\operatorname{cosec} x \cot x + \left(\frac{-12}{x^3} - 3\right)\operatorname{cosec} x$$

9 (a) $y = \frac{x^2}{x+2}$

$$\frac{dy}{dx} = \frac{(x+2)(2x) - x^2}{(x+2)^2}$$

$$= \frac{x^2 + 4x}{(x+2)^2}$$

(b) $y = \frac{x^3 + 4}{x^2 - 6x + 1}$

$$\frac{dy}{dx} = \frac{(x^2 - 6x + 1)(3x^2) - (x^3 + 4)(2x - 6)}{(x^2 - 6x + 1)^2}$$

$$= \frac{3x^4 - 18x^3 + 3x^2 - 2x^4 + 6x^3 - 8x + 24}{(x^2 - 6x + 1)^2}$$

$$= \frac{x^4 - 12x^3 + 3x^2 - 8x + 24}{(x^2 - 6x + 1)^2}$$

(c) $y = \frac{\sin x}{\cos x + 2}$

$$\frac{dy}{dx} = \frac{(\cos x + 2)\cos x - \sin x(-\sin x)}{(\cos x + 2)^2}$$

$$= \frac{\cos^2 x + 2\cos x + \sin^2 x}{(\cos x + 2)^2}$$

$$= \frac{1 + 2\cos x}{(\cos x + 2)^2}$$

(d) $y = \dfrac{\tan x + 1}{\sec x + 3}$

$$\frac{dy}{dx} = \frac{(\sec x + 3)(\sec^2 x) - (\tan x + 1)(\sec x \tan x)}{(\sec x + 3)^2}$$

$$= \frac{\sec^3 x + 3\sec^2 x - \sec x \tan^2 x - \sec x \tan x}{(\sec x + 3)^2}$$

(e) $y = \dfrac{x \cos x}{x^2 + 5}$

$$\frac{dy}{dx} = \frac{(x^2 + 5)(\cos x - x \sin x) - (x \cos x)(2x)}{(x^2 + 5)^2}$$

$$= \frac{x^2\cos x - x^3\sin x + 5\cos x - 5x \sin x - 2x^2\cos x}{(x^2 + 5)^2}$$

$$= \frac{-x^2\cos x - x^3\sin x + 5\cos x - 5x\sin x}{(x^2 + 5)^2}$$

(f) $y = \dfrac{7x}{\sqrt{x + 1}}$

$$\frac{dy}{dx} = \frac{(\sqrt{x + 1})(7) - (7x)\frac{1}{2}(x + 1)^{-\frac{1}{2}}}{(\sqrt{x + 1})^2}$$

$$= \frac{7\sqrt{x + 1} - \dfrac{7x}{2\sqrt{x + 1}}}{x + 1}$$

$$= \frac{7(x + 1) - \frac{7}{2}x}{(x + 1)^{\frac{3}{2}}} = \frac{\frac{7}{2}x + 7}{(x + 1)^{\frac{3}{2}}}$$

(g) $y = \dfrac{x + \sin x}{x + \cos x}$

$$\frac{dy}{dx} = \frac{(x + \cos x)(1 + \cos x) - (x + \sin x)(1 - \sin x)}{(x + \cos x)^2}$$

$$= \frac{\cancel{x} + x\cos x + \cos x + \cos^2 x - \cancel{x} + x\sin x - \sin x + \sin^2 x}{(x + \cos x)^2}$$

$$= \frac{1 + x\cos x + \cos x + x\sin x - \sin x}{(x + \cos x)^2}$$

(h) $y = \dfrac{\cos(3x + 2)}{4x^3 + 2}$

$$\frac{dy}{dx} = \frac{(4x^3 + 2)(-3\sin(3x + 2)) - 12x^2\cos(3x + 2)}{(4x^3 + 2)^2}$$

$$= \frac{-12x^3\sin(3x + 2) - 6\sin(3x + 2) - 12x^2\cos(3x + 2)}{(4x^3 + 2)^2}$$

(i) $y = \dfrac{7x^2 + 2}{x - 4}$

$\dfrac{dy}{dx} = \dfrac{(x-4)(14x) - (7x^2+2)}{(x-4)^2}$

$= \dfrac{14x^2 - 56x - 7x^2 - 2}{(x-4)^2}$

$= \dfrac{7x^2 - 56x - 2}{(x-4)^2}$

(j) $y = \dfrac{x^4}{x^2 - 1}$

$\dfrac{dy}{dx} = \dfrac{(x^2-1)(4x^3) - x^4(2x)}{(x^2-1)^2}$

$= \dfrac{4x^5 - 4x^3 - 2x^5}{(x^2-1)^2}$

$= \dfrac{2x^5 - 4x^3}{(x^2-1)^2} = \dfrac{2x^3(x^2-2)}{(x^2-1)^2}$

10 (a) $y = \cos^3 x$

$\dfrac{dy}{dx} = -3\cos^2 x \sin x$

(b) $y = \sin^3 x$

$\dfrac{dy}{dx} = 3\sin^2 x \cos x$

(c) $y = \tan^4 x$

$\dfrac{dy}{dx} = 4\tan^3 x \sec^2 x$

(d) $y = x^2 \cos^4 x$

$\dfrac{dy}{dx} = 2x\cos^4 x - 4x^2\cos^3 x \sin x$

$=2x \cos^3 x\,[\cos x - 2x \sin x]$

(e) $y = \sec^3(x + 2)$

$\dfrac{dy}{dx} = 3\sec^2(x+2)\sec(x+2)\tan(x+2)$

$= 3\sec^3(x + 2)\tan(x + 2)$

(f) $y = \tan^2(3x + 2)$

$\dfrac{dy}{dx} = 6\tan(3x+2)\sec^2(3x+2)$

(g) $y = (2x + 1)\cot^2 x$

$\dfrac{dy}{dx} = 2\cot^2 x - 2(2x+1)\cot x\,\mathrm{cosec}^2 x$

(h) $y = (4x^2 - x)\sin^2(x+2)$

$\dfrac{dy}{dx} = (8x - 1)\sin^2(x+2) + 2(4x^2 - x)\sin(x+2)\cos(x+2)$

(i) $y = \dfrac{x}{\mathrm{cosec}^2 x + 1}$

$$\frac{dy}{dx} = \frac{(\text{cosec}^2 x + 1) + x\,(2\,\text{cosec}\,x\,\text{cosec}\,x\cot x)}{(\text{cosec}^2 x + 1)^2}$$

$$= \frac{\text{cosec}^2 x + 1 + 2x\,\text{cosec}^2 x\cot x}{(\text{cosec}^2 x + 1)^2}$$

(j) $y = \tan^4 (3x)$

$$\frac{dy}{dx} = 12\tan^3(3x)\sec^2(3x)$$

Try these 13.13

(a) $y = (2x + 1)\cos x^2$

$u = 2x + 1,\ v = \cos x^2$

$$\frac{du}{dx} = 2, \quad \frac{dv}{dx} = -2x\sin x^2$$

$$\frac{dy}{dx} = -2x(2x+1)\sin x^2 + 2\cos x^2$$

$$= (-4x^2 + 4x)\sin x^2 + 2\cos x^2$$

$$\frac{d^2y}{dx^2} = (-8x - 2)\sin x^2 + (-4x^2 - 2x)(2x)\cos x^2 + 2(2x)(-\sin x^2)$$

$$= -2(6x + 1)\sin x^2 - 4x^2(2x + 1)\cos x^2$$

(b) $y = \dfrac{x+2}{2x-1}$

$u = x + 2,\ v = 2x - 1$

$$\frac{du}{dx} = 1, \quad \frac{dv}{dx} = 2.$$

$$\frac{dy}{dx} = \frac{(2x-1)(1) - (x+2)(2)}{(2x-1)^2}$$

$$= \frac{-5}{(2x-1)^2} = -5(2x-1)^{-2}$$

$$\frac{d^2y}{dx^2} = (-5)(-2)(2)(2x-1)^{-3}$$

$$= \frac{20}{(2x-1)^3}$$

$$x = 0, \frac{d^2y}{dx^2} = \frac{20}{(-1)^3} = -20$$

Exercise 13 C

1 $y = (4x + 7)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(4)(4x+7)^{-\frac{1}{2}}$$

$$= 2(4x+7)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(2)(4)(4x+7)^{-\frac{3}{2}}$$

$$= \frac{-4}{(4x+7)^{\frac{3}{2}}} = \frac{-4}{\sqrt{(4x+7)^3}}$$

2 $y = (2x + 3)^{10}$

$$\frac{dy}{dx} = 20(2x+3)^9$$

$$\frac{d^2y}{dx^2} = 360(2x+3)^8$$

3 $y = \frac{1}{(5x-3)^3} = (5x-3)^{-3}$

$$\frac{dy}{dx} = (-3)(5x-3)^{-4}(5) = -15(5x-3)^{-4}$$

$$\frac{d^2y}{dx^2} = (-15)(-4)(5)(5x-3)^{-5}$$

$$= \frac{300}{(5x-3)^5}$$

4 $y = (x + 2) \sin x$

$$\frac{dy}{dx} = \sin x + (x+2)\cos x$$

$$\frac{d^2y}{dx^2} = \cos x + \cos x + (x+2)(-\sin x)$$

$= 2 \cos x - (x + 2) \sin x$

5 $y = \cos(x^2)$

$$\frac{dy}{dx} = -2x\sin(x^2)$$

$$\frac{d^2y}{dx^2} = -2\sin(x^2) + (-2x)(2x)\cos(x^2)$$

$= -2 \sin x^2 - 4x^2 \cos(x^2)$

6 $y = \frac{x^2+2}{x+1}$

$$\frac{dy}{dx} = \frac{(x+1)(2x) - (x^2+2)}{(x+1)^2}$$

$$= \frac{2x^2 + 2x - x^2 - 2}{(x+1)^2}$$

$$= \frac{x^2 + 2x - 2}{(x+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x+1)^2(2x+2) - (x^2 + 2x - 2)(2)(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)[2(x+1) - 2(x^2 + 2x - 2)]}{(x+1)^4}$$

$$= \frac{-2x^2 - 2x + 6}{(x+1)^3}$$

7 $y = \sin(2x^2 + 5x + 1)$

$\frac{dy}{dx} = (4x + 5)\cos(2x^2 + 5x + 1)$

$\frac{d^2y}{dx^2} = (4x + 5)[-(4x + 5)\sin(2x^2 + 5x + 1)] + 4\cos(2x^2 + 5x + 1)$

$= -(4x + 5)^2 \sin(2x^2 + 5x + 1) + 4\cos(2x^2 + 5x + 1)$

8 $y = x^3 \cos^2 x$

$\frac{dy}{dx} = 3x^2 \cos^2 x - 2x^3 \sin x \cos x$

$= 3x^2 \cos^2 x - x^3 \sin 2x$

$\frac{d^2y}{dx^2} = 6x \cos^2 x - 6x^2 \cos x \sin x - 3x^2 \sin 2x - 2x^3 \cos 2x$

$= 6x \cos^2 x - 3x^2 \sin 2x - 3x^2 \sin 2x - 2x^3 \cos 2x$

$= 6x \cos^2 x - 6x^2 \sin 2x - 2x^3 \cos 2x$

9 $y = \cos 3x + \sin 4x$

$\frac{dy}{dx} = -3\sin 3x + 4\cos 4x$

$\frac{d^2y}{dx^2} = -9\cos 3x - 16\sin 4x$

10 $y = \frac{3x - 2}{x - 4}$

$\frac{dy}{dx} = \frac{(x - 4)(3) - (3x - 2)}{(x - 4)^2}$

$= \frac{3x - 12 - 3x + 2}{(x - 4)^2}$

$= \frac{-10}{(x - 4)^2}$

$\frac{d^2y}{dx^2} = \frac{20}{(x - 4)^3}$

11 $y = \frac{3\cos 2x}{x}$

$\frac{dy}{dx} = \frac{-3}{x^2}\cos 2x - \frac{6}{x}\sin 2x$

$\frac{d^2y}{dx^2} = \frac{6}{x^3}\cos 2x + \frac{6}{x^2}\sin 2x + \frac{6}{x^2}\sin 2x - \frac{12}{x}\cos 2x$

$= \frac{6}{x^3}\cos 2x + \frac{12}{x^2}\sin 2x - \frac{12}{x}\cos 2x$

$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy$

$= \cancel{\frac{6}{x^2}\cos 2x} + \cancel{\frac{12}{x}\sin 2x} - \cancel{12\cos 2x} - \cancel{\frac{6}{x^2}\cos 2x} - \cancel{\frac{12}{x}\sin 2x} + \cancel{12\cos 2x}$

$= 0$

12 $y = (x^2 + 1)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(2x)(x^2+1)^{-\frac{1}{2}}=\frac{x}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2}=(x^2+1)^{-\frac{1}{2}}+(x)\left(-\frac{1}{2}\right)(2x)(x^2+1)^{-\frac{3}{2}}$$

$$=(x^2+1)^{-\frac{1}{2}}-x^2(x^2+1)^{-\frac{3}{2}}=\frac{1}{\sqrt{x^2+1}}-\frac{x^2}{(x^2+1)\sqrt{x+1}}$$

$$=\frac{x^2+1-x^2}{(x^2+1)\sqrt{x+1}}$$

$$=\frac{1}{(x^2+1)\sqrt{x+1}}$$

13 $y=2x^2+\frac{4}{x}$

$$\frac{dy}{dx}=4x-\frac{4}{x^2}$$

$$\frac{d^2y}{dx^2}=4+\frac{8}{x^3}$$

$$x^2\frac{d^2y}{dx^2}=4x^2+\frac{8}{x}$$

$$x^2\frac{d^2y}{dx^2}=2y$$

14 $y=\frac{x+2}{x-1}$

$$\frac{dy}{dx}=\frac{(x-1)-(x+2)}{(x-1)^2}$$

$$=\frac{-3}{(x-1)^2}$$

$$\frac{d^2y}{dx^2}=\frac{6}{(x-1)^3}$$

$$3\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)^2(x-1)$$

$$=\frac{18}{(x-1)^3}-2\frac{9(x-1)}{(x-1)^4}=\frac{18}{(x-1)^3}-\frac{18}{(x-1)^3}=0$$

15 $y = \cos x + \sin x$

$$\frac{dy}{dx}=-\sin x+\cos x$$

$$\frac{d^2y}{dx^2}=-\cos x-\sin x$$

$$\frac{d^2y}{dx^2}=-y$$

$$\frac{d^2y}{dx^2}+y=0$$

16 $y = x^2\cos^2 x$

$\frac{dy}{dx} = 2x\cos^2 x - 2x^2 \sin x \cos x$

$= 2x\cos^2 x - x^2 \sin 2x$

$\frac{d^2y}{dx^2} = 2\cos^2 x - 4x\sin x\cos x - 2x^2\cos 2x - 2x\sin 2x$

$x = \frac{\pi}{2},\ \frac{d^2y}{dx^2} = 2\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{2}$

17 $y = \frac{x^2+1}{x^2-1}$

$\frac{dy}{dx} = \frac{(x^2-1)(2x) + (x^2+1)(2x)}{(x^2-1)^2}$

$= \frac{2x^3 - 2x + 2x^3 + 2x}{(x^2-1)^2}$

$= \frac{4x^3}{(x^2-1)^2}$

$\frac{d^2y}{dx^2} = \frac{(x^2-1)^2(12x^2) - (4x^3)(2)(2x)(x^2-1)}{(x^2-1)^4}$

$= \frac{(x^2-1)[12x^2(x^2-1) - 16x^4]}{(x^2-1)^4}$

$= \frac{12x^4 - 12x^2 - 16x^4}{(x^2-1)^3}$

$= \frac{-4x^4 - 12x^2}{(x^2-1)^3}$

When $x = 0,\ \frac{d^2y}{dx^2} = \frac{0}{-1} = 0$

18 $y = \frac{2x}{x+4}$

$\frac{dy}{dx} = \frac{(x+4)(2) - 2x}{(x+4)^2}$

$= \frac{2x + 8 - 2x}{(x+4)^2} = \frac{8}{(x+4)^2} = 8(x+4)^{-2}$

$\frac{d^2y}{dx^2} = -16(x+4)^{-3} = \frac{-16}{(x+4)^3}$

$(x+4)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = \frac{-16(x+4)}{(x+4)^3} + \frac{16}{(x+4)^2}$

$= \frac{-16}{(x+4)^2} + \frac{16}{(x+4)^2} = 0$

19 $y = \frac{1}{x^2 - x} = (x^2 - x)^{-1}$

$\frac{dy}{dx} = -(2x-1)(x^2-x)^{-2}$

$= \dfrac{1-2x}{(x^2-x)^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{(x^2-x)^2(-2)-(1-2x)(2)(2x-1)(x^2-x)}{(x^2-x)^4}$

$= \dfrac{(x^2-x)[(x^2-x)(-2)+2(2x-1)^2]}{(x^2-x)^4}$

$= \dfrac{-2x^2+2x+8x^2-8x+2}{(x^2-x)^3}$

$= \dfrac{6x^2-6x+2}{(x^2-x)^3}$

$x = 2,\quad \dfrac{d^2y}{dx^2} = \dfrac{24\ -12+2}{8}$

$= \dfrac{14}{8} = \dfrac{7}{4}$

20 $y = \dfrac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}(2x)(1+x^2)^{-\frac{3}{2}} = -x(1+x^2)^{-\frac{3}{2}}$

$\dfrac{d^2y}{dx^2} = \dfrac{3}{2}x(2x)(1+x^2)^{-\frac{5}{2}} + (-1)(1+x^2)^{-\frac{3}{2}}$

$= 3x^2(1+x^2)^{-\frac{5}{2}} - (1+x^2)^{-\frac{3}{2}}$

$(1+x^2)\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y$

$= (1+x^2)[3x^2(1+x^2)^{-\frac{5}{2}} - (1+x^2)^{-\frac{3}{2}}] + 3x[-x(1+x^2)^{-\frac{3}{2}}] + (1+x^2)^{-\frac{1}{2}}$

$= \cancel{3x^2(1+x^2)^{-\frac{3}{2}}} - \cancel{(1+x^2)^{-\frac{1}{2}}} - \cancel{3x^2(1+x^2)^{-\frac{3}{2}}} + \cancel{(1+x^2)^{-\frac{1}{2}}}$

$= 0$

Review Exercise 13

1 $f(x) = x + \dfrac{1}{x}$

$f(x+h) = x + h + \dfrac{1}{x+h}$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{x+h+\dfrac{1}{x+h}-\left(x+\dfrac{1}{x}\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{h+\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

$$= \lim_{h \to 0} \frac{h}{h} + \lim_{h \to 0} \frac{\frac{x - (x+h)}{(x+h)(x)}}{h}$$

$$= \lim_{h \to 0} 1 + \lim_{h \to 0} \left[\frac{\frac{-h}{(x+h)(x)}}{h} \right]$$

$$= 1 - \lim_{h \to 0} \frac{1}{(x+h)(x)}$$

$$= 1 - \frac{1}{(x+0)x}$$

$$= 1 - \frac{1}{x^2}$$

2 $f(x) = 2x^2 - 5x + 2$

$f(x+h) = 2(x+h)^2 - 5(x+h) + 2$

$= 2[x^2 + 2xh + h^2] - 5x - 5h + 2$

$= 2x^2 + 4xh + 2h^2 - 5x - 5h + 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{2x^2} + 4xh + 2h^2 - \cancel{5x} - 5h + \cancel{2}) - (\cancel{2x^2} - \cancel{5x} + \cancel{2})}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 5h}{h}$$

$$= \lim_{h \to 0} (4x + 2h - 5) = 4x + 0 - 5$$

$= 4x - 5$

3 $f(x) = x^3 - 2x + 1$

$f(x+h) = (x+h)^3 - 2(x+h) + 1$

$= x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{2x} - 2h + \cancel{1} - (\cancel{x^3} - \cancel{2x} + \cancel{1})}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 2)$$

$= 3x^2 + 3x(0) + 0^2 - 2$

$= 3x^2 - 2$

4 $f(x) = \dfrac{1}{2x+3}$

$f(x+h) = \dfrac{1}{2(x+h)+3}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2(x+h)+3} - \frac{1}{2x+3}}{h}$$

$$= \lim_{h \to 0} \frac{2x+3-(2x+2h+3)}{h(2(x+h)+3)(2x+3)}$$

$$= \lim_{h \to 0} \frac{-2h}{h(2(x+h)+3)(2x+3)}$$

$$= \lim_{h \to 0} \frac{-2}{(2(x+h)+3)(2x+3)}$$

$$= \frac{-2}{(2(x+0)+3)(2x+3)} = \frac{-2}{(2x+3)^2}$$

5 $f(x) = \sin 2x$

$f(x+h) = \sin 2(x+h)$

$= \sin(2x+2h)$

$$f'(x) = \lim_{h \to 0} \frac{\sin(2x+2h) - \sin 2x}{h}$$

$$= \lim_{h \to 0} \frac{\sin 2x \cos 2h + \cos 2x \sin 2h - \sin 2x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sin 2x \cos 2h - \sin 2x}{h} + \frac{\cos 2x \sin 2h}{h} \right)$$

$$= \lim_{h \to 0} \sin 2x \left(\frac{\cos 2h - 1}{h} \right) + \lim_{h \to 0} \cos 2x \left(\frac{\sin 2h}{h} \right)$$

$$= \sin 2x \lim_{h \to 0} \left(\frac{2 \cos 2h - 1}{2h} \right) + \cos 2x \lim_{h \to 0} 2 \left(\frac{\sin 2h}{2h} \right)$$

$= (\sin 2x)(2)(0) + (\cos 2x)(2)(1)$

$= 2 \cos 2x$

OR

$f(x+h) - f(x)$

$= \sin(2x+2h) - \sin 2x$

$$= 2 \cos \left(\frac{2x+2h+2x}{2} \right) \sin \left(\frac{2x+2h-2x}{2} \right)$$

$= 2 \cos(2x+h) \sin h.$

$$\therefore f'(x) = \lim_{h \to 0} \frac{2 \cos(2x+h) \sin h}{h}$$

$$= 2 \lim_{h \to 0} \cos(2x+h) \lim_{h \to 0} \frac{\sin h}{h}$$

$= 2(\cos 2x)(1)$

$= 2 \cos 2x$

6 $f(x) = \cos 2x$

$f(x+h) = \cos(2x+2h)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$= \lim_{h \to 0} \frac{\cos(2x + 2h) - \cos 2x}{h}$

$= \lim_{h \to 0} \frac{\cos 2x \cos 2h - \sin 2x \sin 2h - \cos 2x}{h}$

$= \lim_{h \to 0} \left[\frac{\cos 2x \cos 2h - \cos 2x}{h} - \frac{\sin 2x \sin 2h}{h} \right]$

$= \lim_{h \to 0} 2 \cos 2x \left[\frac{\cos 2h - 1}{2h} \right] - \lim_{h \to 0} (2 \sin 2x) \frac{\sin 2h}{2h}$

$= (2 \cos 2x)(0) - (2 \sin 2x)(1)$

$= -2 \sin 2x$

7 $y = \frac{5x - 2}{3x + 7}$

$\frac{dy}{dx} = \frac{(3x + 7)(5) - (5x - 2)(3)}{(3x + 7)^2}$

$= \frac{15x + 35 - 15x + 6}{(3x + 7)^2}$

$= \frac{41}{(3x + 7)^2} = 41(3x + 7)^{-2}$

$\frac{d^2y}{dx^2} = (41)(-2)(3)(3x + 7)^{-3}$

$= \frac{-246}{(3x + 7)^3}$

8 $y = A \cos 5x + B \sin 5x$

$\frac{dy}{dx} = -5A \sin 5x + 5B \cos 5x$

$\frac{d^2y}{dx^2} = -25A \cos 5x - 25B \sin 5x$

$\frac{d^2y}{dx^2} = -25[A \cos 5x + B \sin 5x]$

$\frac{d^2y}{dx^2} = -25y$

$\frac{d^2y}{dx^2} + 25y = 0$

9 (a) $\frac{d}{dx}(x^2 \sin 5x) = 5x^2 \cos 5x + 2x \sin 5x$

(b) $\frac{d}{dx}[\cos^4(4x)] = 4 \cos^3(4x)(-\sin 4x)(4)$

$= -16 \cos^3 4x \sin 4x$

(c) $\frac{d}{dx}\left[\left(\frac{1 - 2x}{1 + 2x}\right)^2\right] = 2\left(\frac{1 - 2x}{1 + 2x}\right)\left[\frac{(1 + 2x)(-2) - (1 - 2x)(2)}{(1 + 2x)^2}\right]$

$= 2\left(\frac{1 - 2x}{1 + 2x}\right)\left(\frac{-4}{(1 + 2x)^2}\right) = \frac{-8(1 - 2x)}{(1 + 2x)^3}$

10 (a) $y = (2x + 1)\tan(4x + 5)$

$$\frac{dy}{dx} = (2x+1)[4\sec^2(4x+5)] + 2\tan(4x+5)$$

$$= 4(2x+1)\sec^2(4x+5) + 2\tan(4x+5)$$

(b) $y = \sin(2x+3) + x\tan 4x$

$$\frac{dy}{dx} = 2\cos(2x+3) + \tan 4x + 4x\sec^2(4x)$$

(c) $y = \sin^3(x^2+4x+2)$

$$\frac{dy}{dx} = 3\sin^2(x^2+4x+2)\cos(x^2+4x+2)[2x+4]$$

$$= (6x+12)\sin^2(x^2+4x+2)\cos(x^3+4x+2)$$

(d) $x = \dfrac{1+\cos 2\theta}{\sin 2\theta}$

$$\frac{dx}{d\theta} = \frac{\sin 2\theta(-2\sin 2\theta) - (1+\cos 2\theta)(2\cos 2\theta)}{(\sin 2\theta)^2}$$

$$= \frac{-2\sin^2(2\theta) - 2\cos 2\theta - 2\cos^2 2\theta}{\sin^2 2\theta}$$

$$= \frac{-2[\sin^2(2\theta) + \cos^2 2\theta] - 2\cos 2\theta}{\sin^2 2\theta}$$

$$= \frac{-2-2\cos 2\theta}{\sin^2 2\theta} = \frac{-2(1+\cos 2\theta)}{\sin^2 2\theta} = \frac{-4\cos^2\theta}{4\sin^2\theta\cos^2\theta} = -\frac{1}{\sin^2\theta} = -\operatorname{cosec}^2\theta$$

(e) $t = \tan^2\theta\sin^2\theta$

$$\frac{dt}{d\theta} = \tan^2\theta[2\sin\theta\cos\theta] + \sin^2\theta(2\tan\theta\sec^2\theta)$$

$$= 2\tan^2\theta\sin\theta\cos\theta + 2\frac{\sin^2\theta}{\cos^2\theta}\tan\theta$$

$$= 2\tan^2\theta\sin\theta\cos\theta + 2\tan^3\theta$$

$$= \tan^2\theta(2\sin\theta\cos\theta + 2\tan\theta)$$

$$= \tan^2\theta(\sin 2\theta + 2\tan\theta)$$

(f) $r = \sin^2 x + 2\cos^3 x$

$$\frac{dr}{dx} = 2\sin x\cos x + 3(2)\cos^2 x(-\sin x)$$

$$= 2\sin x\cos x - 6\sin x\cos^2 x$$

$$= 2\sin x\cos x[1 - 3\cos x] = \sin 2x(1 - 3\cos x)$$

11 $y = x^3\cos(2x^2+\pi)$

$$\frac{dy}{dx} = x^3[4x(-\sin(2x^2+\pi))] + 3x^2\cos(2x^2+\pi)$$

$$= -4x^4\sin(2x^2+\pi) + 3x^2\cos(2x^2+\pi)$$

$$x = 0,\ \frac{dy}{dx} = 0$$

12 $y = \sqrt{4x-3}$

$$y = (4x-3)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(4)(4x-3)^{-\frac{1}{2}}$$

$= 2\,(4x - 3)^{-\frac{1}{2}}$

$\dfrac{d^2y}{dx^2} = -\dfrac{1}{2}(2)\,(4)\,(4x-3)^{-\frac{3}{2}}$

$= \dfrac{-4}{\left(\sqrt{4x-3}\right)^3} = \dfrac{-4}{y^3}$

13 $y = \dfrac{x}{\sqrt{x^2+32}}$

$\dfrac{dy}{dx} = \dfrac{(x^2+32)^{\frac{1}{2}}\,(1) - \left(\frac{1}{2}\right)(2x)\,(x^2+32)^{-\frac{1}{2}}(x)}{\left(\sqrt{x^2+32}\right)^2}$

$= \dfrac{\sqrt{x^2+32} - \dfrac{x^2}{\sqrt{x^2+32}}}{x^2+32}$

$= \dfrac{\dfrac{x^2+32-x^2}{\sqrt{x^2+32}}}{x^2+32}$

$= \dfrac{32}{(x^2+32)\,\sqrt{x^2+32}}$

$= \dfrac{32}{(x^2+32)^{\frac{3}{2}}}$

14 $y = \dfrac{\cos x}{1-\sin x}$

$\dfrac{dy}{dx} = \dfrac{(1-\sin x)\,(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$= \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$

$= \dfrac{1-\sin x}{(1-\sin x)^2} = \dfrac{1}{1-\sin x}$

$x = \pi/4,\ \dfrac{dy}{dx} = \dfrac{1}{1-\sin \pi/4} = \dfrac{1}{1-\dfrac{\sqrt{2}}{2}}$

$= \dfrac{2}{2-\sqrt{2}}$

$= \dfrac{2}{2-\sqrt{2}} \times \dfrac{2+\sqrt{2}}{2+\sqrt{2}}$

$= \dfrac{2(2+\sqrt{2})}{2} = 2+\sqrt{2}$

15 $y = \dfrac{1+\cos x}{1-\sin x}$

$$\frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - (1+\cos x)(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1 - \sin x + \cos x}{(1-\sin x)^2}$$

When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{1 - \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)}{\left(1 - \sin\frac{\pi}{3}\right)^2}$

$$= \frac{1 - \frac{\sqrt{3}}{2} + \frac{1}{2}}{\left(1 - \frac{\sqrt{3}}{2}\right)^2} = \frac{\frac{3}{2} - \frac{\sqrt{3}}{2}}{1 - \sqrt{3} + \frac{3}{4}} = \frac{\frac{3-\sqrt{3}}{2}}{\frac{7-4\sqrt{3}}{4}}$$

$$= \frac{2(3-\sqrt{3})}{7-4\sqrt{3}} = \frac{2(3-\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})} = \frac{2(21+12\sqrt{3}-7\sqrt{3}-12)}{49-48}$$

$$= 2(9+5\sqrt{3})$$

16 $y = \frac{x}{1-5x}$

(a) $\frac{dy}{dx} = \frac{(1-5x)(1) - x(-5)}{(1-5x)^2}$

$$= \frac{1-5x+5x}{(1-5x)^2}$$

$$= \frac{1}{(1-5x)^2}$$

(b) Since $\frac{dy}{dx} = \frac{1}{(1-5x)^2}$

$$x^2\frac{dy}{dx} = \frac{x^2}{(1-5x)^2}$$

$$\Rightarrow x^2\frac{dy}{dx} = \left(\frac{x}{1-5x}\right)^2$$

$$\Rightarrow x^2\frac{dy}{dx} = y^2$$

(c) $\frac{dy}{dx} = (1-5x)^{-2}$

$$\frac{d^2y}{dx^2} = -2(1-5x)^{-3}(-5)$$

$$= \frac{10}{(1-5x)^3}$$

$$x^2\frac{d^2y}{dx^2} = \frac{10x^2}{(1-5x)^3} = 10\left(\frac{x}{1-5x}\right)^2\left(\frac{1}{1-5x}\right)$$

$$x^2\frac{d^2y}{dx^2} = 10\,y^2\left(\frac{y}{x}\right)$$

$$x^2\frac{d^2y}{dx^2} = \frac{10y^3}{x}$$

17 (a) $y = \dfrac{x+1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{-2}{(x-1)^2}$$

(b) Since $\dfrac{dy}{dx} = \dfrac{-2}{(x-1)^2}$

$$\Rightarrow (x+1)^2\frac{dy}{dx} = -2\frac{(x+1)^2}{(x-1)^2}$$

$$= -2\left(\frac{x+1}{x-1}\right)^2$$

$$=-2y^2$$

$$\therefore (x+1)^2\frac{dy}{dx} = -2y^2$$

(c) $y = \dfrac{x+1}{x-1}$

$$\frac{dy}{dx} = -2(x-1)^{-2} = \frac{-2}{(x-1)^2}$$

$$\frac{d^2y}{dx^2} = 4(x-1)^{-3} = \frac{4}{(x-1)^3}$$

$$(x+1)^2\frac{d^2y}{dx^2} + 2(x+2y+1)\frac{dy}{dx} = \frac{4(x+1)^2}{(x-1)^3} + 2\left[x + \frac{2(x+1)}{x-1} + 1\right]\left[\frac{-2}{(x-1)^2}\right]$$

$$= \frac{4(x+1)^2}{(x-1)^3} - \frac{4x}{(x-1)^2} - \frac{8(x+1)}{(x-1)^3} - \frac{4}{(x-1)^2}$$

$$= \frac{4(x+1)^2 - 4x(x-1) - 8(x+1) - 4(x-1)}{(x-1)^3}$$

$$= \frac{\cancel{4x^2} + \cancel{8x} + \cancel{4} - \cancel{4x^2} + \cancel{4x} - \cancel{8x} - \cancel{8} - \cancel{4x} + \cancel{4}}{(x-1)^3}$$

$$= \frac{0}{(x-1)^3} = 0$$

18 $y = \dfrac{x^2}{2-3x^2}$

$$\frac{dy}{dx} = \frac{(2-3x^2)(2x) - x^2(-6x)}{(2-3x^2)^2}$$

$$= \frac{4x - 6x^3 + 6x^3}{(2 - 3x^2)^2}$$

$$= \frac{4x}{(2 - 3x^2)^2}$$

Since $\frac{dy}{dx} = \frac{4x}{(2 - 3x^2)^2}$

$$x^3 \frac{dy}{dx} = \frac{4x^4}{(2 - 3x^2)^2}$$

$$\Rightarrow x^3 \frac{dy}{dx} = 4\left[\frac{x^2}{2 - 3x^2}\right]^2$$

$$x^3 \frac{dy}{dx} = 4y^2$$

19 (a) $y = \cos^3 x \sin x$

$$\frac{dy}{dx} = \cos^3 x \cos x + \sin x\,[(3\cos^2 x)(-\sin x)]$$

$$= \cos^4 x - 3\cos^2 x \sin^2 x = \cos^2 x\,[\cos^2 x - 3\sin^2 x]$$

(b) When $x = \frac{\pi}{4}$, $\frac{dy}{dx} = \left(\cos\frac{\pi}{4}\right)^4 - 3\cos^2\frac{\pi}{4}\sin^2\frac{\pi}{4}$

$$= \left(\frac{\sqrt{2}}{2}\right)^4 - 3\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right)^2$$

$$= \frac{4}{16} - 3\left(\frac{4}{16}\right)$$

$$= \frac{1}{4} - \frac{3}{4}$$

$$= -\frac{1}{2}$$

20 (a) $\theta = \sin(2t - \pi)^{\frac{1}{2}}$

$$\frac{d\theta}{dt} = \frac{1}{2}(2t - \pi)^{-\frac{1}{2}}(2)\cos(2t - \pi)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2t - \pi}}\cos\left(\sqrt{2t - \pi}\right)$$

(b) $\theta = t\sqrt{4t^2 - 3t + 2}$

$$\frac{d\theta}{dt} = t\left[\frac{1}{2}(4t^2 - 3t + 2)^{-\frac{1}{2}}(8t - 3)\right] + \sqrt{4t^2 - 3t + 2}$$

$$= \frac{\frac{1}{2}t(8t - 3)}{\sqrt{4t^2 - 3t + 2}} + \sqrt{4t^2 - 3t + 2}$$

$$= \frac{\frac{1}{2}t\,(8t - 3) + 4t^2 - 3t + 2}{\sqrt{4t^2 - 3t + 2}}$$

$$= \frac{4t^2 - \frac{3}{2}t + 4t^2 - 3t + 2}{\sqrt{4t^2 - 3t + 2}} = \frac{8t^2 - \frac{9}{2}t + 2}{\sqrt{4t^2 - 3t + 2}}$$

(c) $\theta = \frac{1 + \cos 2t}{1 - \sin 2t}$

$$\frac{d\theta}{dt} = \frac{(1 - \sin 2t)(-2\sin 2t) - (1 + \cos 2t)(-2\cos 2t)}{(1 - \sin 2t)^2}$$

$$= \frac{-2\sin 2t + 2\sin^2 2t + 2\cos 2t + 2\cos^2 2t}{(1 - \sin 2t)^2}$$

$$= \frac{2(\sin^2 2t + \cos^2 2t) - 2\sin 2t + 2\cos 2t}{(1 - \sin 2t)^2}$$

$$= \frac{2 - 2\sin 2t + 2\cos 2t}{(1 - \sin 2t)^2}$$

21 $y = (1 + \cos x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(1 + \cos x)^{-\frac{1}{2}}(-\sin x)$$

$$= -\frac{1}{2}\sin x(1 + \cos x)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\sin x\,(1 + \cos x)^{-\frac{3}{2}}(-\sin x) - \frac{1}{2}\cos x(1 + \cos x)^{-\frac{1}{2}}$$

$$= -\frac{1}{4}\sin^2 x(1 + \cos x)^{-\frac{3}{2}} - \frac{1}{2}\cos x(1 + \cos x)^{-\frac{1}{2}}$$

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2$$

$$= 2(1 + \cos x)^{\frac{1}{2}}\left[-\frac{1}{4}\sin^2 x\,(1 + \cos x)^{-\frac{3}{2}} - \frac{1}{2}\cos x(1 + \cos x)^{-\frac{1}{2}}\right] + 2\left[-\frac{1}{2}\sin x(1 + \cos x)^{-\frac{1}{2}}\right]^2 + [(1 + \cos x)^{\frac{1}{2}}]^2$$

$$= -\frac{1}{2}\sin^2 x(1 + \cos x)^{-1} - \cos x + \frac{1}{2}\sin^2 x(1 + \cos x)^{-1} + 1 + \cos x$$

$= 1$