Chapter 11 Vectors in Three Dimensions ($\mathbb{R}^3$)

Exercise 11A

1 (a) $\begin{pmatrix}1\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\2\\4\end{pmatrix} = -1+4+12 = 15$

(b) $\begin{pmatrix}0\\1\\4\end{pmatrix}.\begin{pmatrix}2\\1\\5\end{pmatrix} = 0+1+20 = 21$

(c) $\begin{pmatrix}4\\2\\-3\end{pmatrix}.\begin{pmatrix}-1\\2\\1\end{pmatrix} = -4+4-3 = -3$

(d) $\begin{pmatrix}4\\2\\2\end{pmatrix}.\begin{pmatrix}3\\4\\-4\end{pmatrix} = 12+8-8 = 12$

2 (a) $\left|\overrightarrow{OA}\right| = \sqrt{4^2+2^2+1^2} = \sqrt{21}$

(b) $\left|\overrightarrow{OB}\right| = \sqrt{3^2+2^2+(-2)^2} = \sqrt{17}$

(c) $\left|\overrightarrow{OC}\right| = \sqrt{4^2+(-1)^2+(-2)^2} = \sqrt{21}$

3 (a) $\begin{pmatrix}1\\2\\-1\end{pmatrix}.\begin{pmatrix}3\\2\\2\end{pmatrix} = 3+4-2 = 5$

Not Perpendicular

(b) $\begin{pmatrix}1\\4\\2\end{pmatrix}.\begin{pmatrix}0\\1\\-2\end{pmatrix} = 0+4-4 = 0$

Perpendicular

(c) $\begin{pmatrix}2\\3\\-2\end{pmatrix}.\begin{pmatrix}-2\\2\\1\end{pmatrix} = -4+6-2 = 0$

Perpendicular

(d) $\begin{pmatrix}4\\2\\5\end{pmatrix}.\begin{pmatrix}0\\1\\-1\end{pmatrix} = 0+2-5 = -3$

Not perpendicular

4 (a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-4)^2 + (-3)^2 + 2^2} = \sqrt{29}$$

A unit vector in the direction of $\overrightarrow{AB}$ is

$$\begin{pmatrix} -4/\sqrt{29} \\ -3/\sqrt{29} \\ 2/\sqrt{29} \end{pmatrix}$$

(b) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \\ 0 \end{pmatrix}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{(-3)^2 + (-5)^2} = \sqrt{34}$$

A unit vector in the direction of $\overrightarrow{AB}$ is

$$\begin{pmatrix} -3/\sqrt{34} \\ -5/\sqrt{34} \\ 0 \end{pmatrix}$$

(c) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$$= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{(1)^2 + (-2)^2 + (-2)^2}$$

$$= \sqrt{9} = 3$$

A unit vector in the direction of $\overrightarrow{AB}$ is

$$\begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

5 (a) Let θ be the angle between a and b

$$\cos\theta = \frac{\begin{pmatrix}2\\4\\-1\end{pmatrix}.\begin{pmatrix}3\\2\\2\end{pmatrix}}{\left|\begin{pmatrix}2\\4\\-1\end{pmatrix}\right|\left|\begin{pmatrix}3\\2\\2\end{pmatrix}\right|}$$

$$= \frac{6+4-2}{\sqrt{21}\sqrt{17}}$$

$$= \frac{8}{\sqrt{21}\sqrt{17}}$$

$\Rightarrow \theta = 65°$

(b) a.c = 0

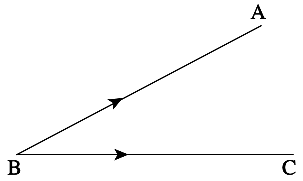
$$\Rightarrow \begin{pmatrix}1\\p\\2p-1\end{pmatrix}.\begin{pmatrix}2\\4\\-1\end{pmatrix} = 0$$

$\Rightarrow 2 + 4p - 2p + 1 = 0$

$2p = -3$

$$p = \frac{-3}{2}$$

6



$$\overrightarrow{BA} = \begin{pmatrix}2\\1\\-4\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-5\end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\\-2\end{pmatrix}$$

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}$$

$$= \frac{\begin{pmatrix}-1\\-1\\-5\end{pmatrix}.\begin{pmatrix}-2\\-1\\-2\end{pmatrix}}{\left|\begin{pmatrix}-1\\-1\\-5\end{pmatrix}\right|\left|\begin{pmatrix}-2\\-1\\-2\end{pmatrix}\right|}$$

$$= \frac{2+1+10}{\sqrt{27}\,\sqrt{9}}$$

$$= \frac{13}{9\sqrt{3}}$$

$$\Rightarrow \theta = 33.5°$$

7 $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \ \overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$$\overrightarrow{AC} = 2\,\overrightarrow{AB}$$

$$\Rightarrow \overrightarrow{OC} - \overrightarrow{OA} = 2\left[\overrightarrow{OB} - \overrightarrow{OA}\right]$$

$$\overrightarrow{OC} = 2\,\overrightarrow{OB} - 2\,\overrightarrow{OA} + \overrightarrow{OA}$$

$$= 2\,\overrightarrow{OB} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$\left|\overrightarrow{OC}\right| = \sqrt{9} = 3$$

$\therefore$ A unit vector in the direction of $\overrightarrow{OC}$ is $\begin{pmatrix} 0/3 \\ 0/3 \\ 3/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

8 (a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

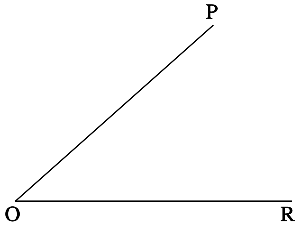
$$= \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(1)^2 + (-1)^2 + 2^2} = \sqrt{6}$$

A unit vector in the direction of PQ is $\begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$

(b)



$\overrightarrow{OP} \cdot \overrightarrow{OR} = 0$

$\Rightarrow \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ a \end{pmatrix} = 0$

$\Rightarrow -6 + 2 + 4a = 0$

$4a = 4$

$a = 1$

(c) $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP}$

$= \begin{pmatrix} -1 \\ 3 \\ b \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 0 \\ b-2 \end{pmatrix}$

$\left| \overrightarrow{PS} \right| = 5$

$\Rightarrow \sqrt{(-3)^2 + (0)^2 + (b-2)^2} = 5$

$\Rightarrow 9 + 0 + (b-2)^2 = 25$

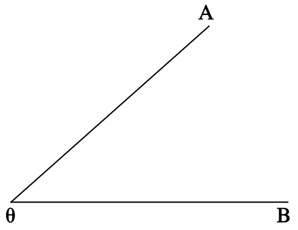
$(b-2)^2 = 16$

$b - 2 = \pm 4$

$b = 6, -2$

9 (a) $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$



$$\cos\theta = \frac{\begin{pmatrix}-2\\4\\2\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}}{\left|\begin{pmatrix}-2\\4\\2\end{pmatrix}\right|\left|\begin{pmatrix}3\\1\\1\end{pmatrix}\right|}$$

$$= \frac{-6+4+2}{\sqrt{24}\,\sqrt{11}}$$

$= 0$

$\Rightarrow \theta = 90°$

(b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \begin{pmatrix}3\\1\\p\end{pmatrix} - \begin{pmatrix}-2\\4\\2\end{pmatrix}$$

$$= \begin{pmatrix}5\\-3\\p-2\end{pmatrix}$$

$\left|\overrightarrow{AB}\right| = 8$

$\Rightarrow \sqrt{(5)^2 + (-3)^2 + (p-2)^2} = 8$

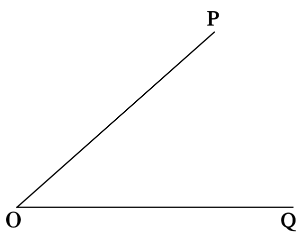
$\Rightarrow 34 + (p-2)^2 = 64$

$(p-2)^2 = 30$

$p - 2 = \pm\sqrt{30}$

$p = 2 \pm \sqrt{30}$

10 (a) $\overrightarrow{OP} = \begin{pmatrix}3\\2\\5\end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix}5\\-4\\5\end{pmatrix}$



$$\cos\theta = \frac{\begin{pmatrix}3\\2\\5\end{pmatrix}\cdot\begin{pmatrix}5\\-4\\5\end{pmatrix}}{\left|\begin{pmatrix}3\\2\\5\end{pmatrix}\right|\left|\begin{pmatrix}5\\-4\\5\end{pmatrix}\right|}$$

$$= \frac{15-8+25}{\sqrt{38}\,\sqrt{66}}$$

$$= \frac{32}{\sqrt{38}\,\sqrt{66}}$$

$\theta = 50.3°$

(b) Since A is on OQ $\Rightarrow \overrightarrow{OA} = \lambda \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix}$

$$= \begin{pmatrix} 5\lambda \\ -4\lambda \\ 5\lambda \end{pmatrix}$$

$$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 5\lambda \\ -4\lambda \\ 5\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5\lambda - 3 \\ -4\lambda - 2 \\ 5\lambda - 5 \end{pmatrix}$$

$$\overrightarrow{PA} \cdot \overrightarrow{OQ} = 0$$

$$\Rightarrow \begin{pmatrix} 5\lambda - 3 \\ -4\lambda - 2 \\ 5\lambda - 5 \end{pmatrix} . \begin{pmatrix} 5 \\ -4 \\ 5 \end{pmatrix} = 0$$

$$\Rightarrow 25\lambda + 16\lambda + 25\lambda - 15 + 8 - 25 = 0$$

$$66\lambda = 32$$

$$\lambda = \frac{16}{33}$$

Exercise 11B

1 (a) $r = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$

(b) $r = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \lambda \in \mathbb{R}$

(c) $r = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}, t \in \mathbb{R}$

2 (a) Using r = a + λ (b – a), $\lambda \in \mathbb{R}$

$$r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \left[\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right], \lambda \in \mathbb{R}$$

$$r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

(b) $r = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\left[\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}\right]$

$r = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, t \in \mathbb{R}$

(c) $r = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} + s\left(\begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}\right)$

$r = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} + s\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, s \in \mathbb{R}$

3 Using r. n = a. n.

(a) $r.\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}.\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$

$r.\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = 2 + 16 - 2 = 16$

$r.\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = 16$

$\Rightarrow x + 4y + z = 16$

(b) $r.\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}.\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$

$\Rightarrow r.\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = 8 + 4 + 10 = 22 \Rightarrow r.\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 11$

$4x + 2y + 2z = 22$
$2x + y + z = 11$

(c) $r.\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 3 \end{pmatrix}.\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$

$r.\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} = -6 - 12 = -18 \Rightarrow r.\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = 18$

$-x - 4z = -18$
$x + 4z = 18$

4 vector equation of l:

$$r = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -\lambda \\ 2\lambda \\ \lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-\lambda \\ 1+2\lambda \\ 3+\lambda \end{pmatrix}$$

$$\left.\begin{aligned} \Rightarrow x &= 2-\lambda \\ y &= 1+2\lambda \\ z &= 3+\lambda \end{aligned}\right\} \lambda \in \mathbb{R} \quad \text{parametric equations}$$

$\lambda = 2 - x$

$\lambda = \dfrac{y-1}{2}$

$\lambda = z - 3$

$\therefore 2 - x = \dfrac{y-1}{2} = z - 3 \quad \rightarrow$ cartesian equation

5 $\lambda = \dfrac{x-2}{4} \Rightarrow 4\lambda = x - 2$

$x = 2 + 4\lambda$

$\lambda = \dfrac{y-3}{5} \Rightarrow 5\lambda = y - 3$

$y = 5\lambda + 3$

$\lambda = \dfrac{2-z}{3} \Rightarrow 3\lambda = 2 - z$

$z = 2 - 3\lambda$

$\therefore\ x = 2 + 4\lambda$

$y = 3 + 5\lambda$

$z = 2 - 3\lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$$

$$\therefore r = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R}$$

6 $x = 2 + \lambda$

$y = 3 + 4\lambda$

$z = 2 + 2\lambda$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$$\Rightarrow r = \begin{pmatrix}2\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\2\end{pmatrix}, \lambda \in \mathbb{R}$$

7 (a) $r = \begin{pmatrix}2\\-3\\4\end{pmatrix} + t\begin{pmatrix}4\\5\\3\end{pmatrix}, t \in \mathbb{R}$

(b) $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2+4t\\-3+5t\\4+3t\end{pmatrix}$

$$\left.\begin{aligned}\Rightarrow x &= 2+4t\\ y &= -3+5t\\ z &= 4+3t\end{aligned}\right\} t \in \mathbb{R},\ \text{parametric equations}$$

(c) $x = 2+4t \Rightarrow t = \dfrac{x-2}{4}$

$y = -3+5t \Rightarrow t = \dfrac{y+3}{5}$

$z = 4+3t \Rightarrow t = \dfrac{z-4}{3}$

$\therefore \dfrac{x-2}{4} = \dfrac{y+3}{5} = \dfrac{z-4}{3}$ is the cartesian equation

8 A vector parallel to l_i is: $-i - 2j - k$
A point on l_1 is : $2i - 5j - k$
A vector parallel to l_2 is $-3i + 5j - k$
A point on l_2 is $4i + j + 2k$

$$\frac{x+2}{3} = \frac{y-1}{4} = \frac{z+2}{-1}$$

$$\Rightarrow \frac{x-(-2)}{3} = \frac{y-1}{4} = \frac{z-(-2)}{-1}$$

A point on l_3 is $\begin{pmatrix}-2\\1\\-2\end{pmatrix}$ and a vector parallel to l_3 is $\begin{pmatrix}3\\4\\-1\end{pmatrix}$

9 $\overrightarrow{OC} = \begin{pmatrix}6\\4\\5\end{pmatrix}, \overrightarrow{OD} = \begin{pmatrix}12\\p\\q\end{pmatrix}$

(a) $r = \begin{pmatrix}6\\4\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\0\end{pmatrix}$

Equation of CD : $r = \begin{pmatrix}6\\4\\5\end{pmatrix} + \lambda\left[\begin{pmatrix}12\\p\\q\end{pmatrix} - \begin{pmatrix}6\\4\\5\end{pmatrix}\right]$

$$r = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ p-4 \\ q-5 \end{pmatrix}$$

$$\therefore r = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} + 6\lambda \begin{pmatrix} 1 \\ \frac{p-4}{6} \\ \frac{q-5}{6} \end{pmatrix}$$

Comparing the direction vectors:

$$\frac{p-4}{6} = -1 \Rightarrow p = -6 + 4 = -2$$

$$\frac{q-5}{6} = 0 \Rightarrow q = 5$$

(b) $$r = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6+\lambda \\ 4-\lambda \\ 5 \end{pmatrix}$$

Since A is on the line

$$\overrightarrow{OA} = \begin{pmatrix} 6+\lambda \\ 4-\lambda \\ 5 \end{pmatrix}$$

Since $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{CD}$

$$\overrightarrow{OA} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 6+\lambda \\ 4-\lambda \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow 6 + \lambda - 4 + \lambda = 0$

$2\lambda + 2 = 0$

$\lambda = -1$

$$\therefore \overrightarrow{OA} = \begin{pmatrix} 6-1 \\ 4-(-1) \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

10 l: $r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$

$$\overrightarrow{OP} = \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ 2+3\lambda \end{pmatrix}$$

Since $\overrightarrow{OP}$ is perpendicular to l

$$\Rightarrow \overrightarrow{OP} . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ 2+3\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$\Rightarrow 2 - 1 + 6 + 4\lambda + \lambda + 9\lambda = 0$$

$$14\lambda = -7$$

$$\lambda = \frac{-1}{2}$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} 1+2\left(-\frac{1}{2}\right) \\ -1-\frac{1}{2} \\ 2+3\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -3/2 \\ 1/2 \end{pmatrix}$$

Since Q is on l:

$$\overrightarrow{OQ} = \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ 2+3\lambda \end{pmatrix}$$

$$\left|\overrightarrow{OQ}\right| = 4$$

$$\Rightarrow \sqrt{(1+2\lambda)^2 + (-1+\lambda)^2 + (2+3\lambda)^2} = 4$$

$$\Rightarrow (1+2\lambda)^2 + (-1+\lambda)^2 + (2+3\lambda)^2 = 4^2$$

$$\Rightarrow 1 + 4\lambda + 4\lambda^2 + 1 - 2\lambda + \lambda^2 + 4 + 12\lambda + 9\lambda^2 = 16$$

$$\Rightarrow 14\lambda^2 + 14\lambda - 10 = 0$$

$$7\lambda^2 + 7\lambda - 5 = 0$$

$$\lambda = \frac{-7 \pm \sqrt{49 - (7)(-5)4}}{14}$$

$$= \frac{-7 \pm \sqrt{189}}{14}$$

$$= \frac{-7 \pm 3\sqrt{21}}{14}$$

11 $$r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2s \\ 1+2s \\ 3s \end{pmatrix}$$

$$r = \begin{pmatrix} -9 \\ 36 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -9+5t \\ 36+2t \\ 1+5t \end{pmatrix}$$

$$\therefore \begin{pmatrix} -2s \\ 1+2s \\ 3s \end{pmatrix} = \begin{pmatrix} -9+5t \\ 36+2t \\ 1+5t \end{pmatrix}$$

$\Rightarrow -2s = -9 + 5t$ [1]
$1 + 2s = 36 + 2t$ [2]
$3s = 1 + 5t$ [3]
$[3] - [1] \Rightarrow 5s = 10$
$s = 2$
Subst. into [1] $\Rightarrow -4 = -9 + 5t$
$t = 1$
Subst. $s = 2$, $t = 1$ into [3] $\Rightarrow$ $3(2) = 1 + 5(1)$
$6 = 6$
Since all three equations are satisfied by $s = 2$ and $t = 1$ $\Rightarrow$ the lines intersect.

Point of intersection: Substituting $s = 2$ into $\begin{pmatrix} -2s \\ 1+2s \\ 3s \end{pmatrix}$

We get $\begin{pmatrix} -2(2) \\ 1+2(2) \\ 3(2) \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}$

12 Using r . n = a. n, we have

$$r\,.\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} . \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow r.\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = 4 - 3$$

$$r.\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = 1$$

$\Rightarrow 2y + 3z = 1$

13 $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$

Equation of the plane:

$$r\,.\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

$$\Rightarrow r\,.\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 8 + 6 - 4$$

$$r \cdot \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 10$$

$$\text{Now} \left| \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \right| = \sqrt{2^2 + 3^2 + (-2)^2} = \sqrt{17}$$

$$\therefore r \cdot \begin{pmatrix} 2/\sqrt{17} \\ 3/\sqrt{17} \\ -2/\sqrt{17} \end{pmatrix} = \frac{10}{\sqrt{17}}$$

Which is of the form $r.\hat{n} = d$

$\therefore$ The distance from the origin to the plane:

$$\frac{10}{\sqrt{17}} = \frac{10\sqrt{17}}{17}$$

14 $\overrightarrow{OP} = 2i + j + k$, $\overrightarrow{OQ} = -32i + 4j + 2k$, $\overrightarrow{OR} = 2i + j + 4k$

(a) Equation of PQ:

$$r = \overrightarrow{OP} + t\left[\overrightarrow{OQ} - \overrightarrow{OP}\right], t \in \mathbb{R}$$

$$\therefore r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\left[\begin{pmatrix} -32 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right]$$

$$r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix}, t \in \mathbb{R}$$

(b) Since PQ is perpendicular to the plane,

$\begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix}$ is a vector perpendicular to the plane

Using r . n = a . n

$$\Rightarrow r \cdot \begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow r \cdot \begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix} = -68 + 3 + 4$$

$$r \cdot \begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix} = -61$$

The equation of the plane is $r \cdot \begin{pmatrix} -34 \\ 3 \\ 1 \end{pmatrix} = -61$

15 A vector parallel to l_1 is $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

A vector parallel to l_2 is $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

Let θ be the angle between l_1 and l_2:

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}\right|}$$

$$= \frac{6-3+16}{\sqrt{29}\sqrt{26}} = \frac{19}{\sqrt{29}\sqrt{26}}$$

θ = 46.2°, No

16 $r = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(a) $\begin{pmatrix} p \\ q \\ 3 \end{pmatrix} = \begin{pmatrix} -1+t \\ 1+2t \\ 2t \end{pmatrix} \Rightarrow 3 = 2t$

$t = \frac{3}{2}$

$\therefore\ p = -1 + \frac{3}{2} = \frac{1}{2}$

$q = 1 + 2\left(\frac{3}{2}\right) = 4$

(b) $t = 2,\ \overrightarrow{OB} = \begin{pmatrix} -1+2 \\ 1+2(2) \\ 2(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$

(c) $\overrightarrow{OC} = \begin{pmatrix} -1+t \\ 1+2t \\ 2t \end{pmatrix}$

Since $\overrightarrow{OC}$ is perpendicular to l

$\Rightarrow \overrightarrow{OC} . \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} -1+t \\ 1+2t \\ 2t \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0$$

$\Rightarrow -1 + t + 2 + 4t + 4t = 0$

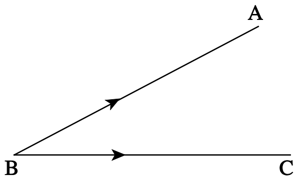
$9t + 1 = 0$

$$t = -\frac{1}{9}$$

$$\therefore \overrightarrow{OC} = \begin{pmatrix} -1-\frac{1}{9} \\ 1+2\left(-\frac{1}{9}\right) \\ 2\left(-\frac{1}{9}\right) \end{pmatrix} = \begin{pmatrix} -10/9 \\ 7/9 \\ -2/9 \end{pmatrix}$$

Review Exercise 11

1



$$\overrightarrow{BA} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

Let θ be angle ABC:

$$\cos\theta = \frac{\begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}}{\left|\begin{pmatrix} -4 \\ -5 \\ -6 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}\right|} = \frac{-12-10-12}{\sqrt{77}\sqrt{17}} = \frac{-34}{\sqrt{77}\sqrt{17}}$$

$\theta = 160°$

2 $$r . \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 6$$

$$\left|\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$r\,.\begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}=\frac{6}{3}=2$$

The equation is of the form $r\,.\,\hat{n}=d$

$\therefore d=2$

The distance from the origin to the plane is 2 units

3 $r=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$

$$r=\begin{pmatrix}-4\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$\begin{pmatrix}1-\lambda\\2\lambda\\1+3\lambda\end{pmatrix}=\begin{pmatrix}-4+\mu\\2+3\mu\\-1-2\mu\end{pmatrix}$$

$\Rightarrow 1-\lambda=-4+\mu$ [1]

$2\lambda=2+3\mu$ [2]

$1+3\lambda=-1-2\mu$ [3]

$[1]\times 2\Rightarrow 2-2\lambda=-8+2\mu$ [4]

$[2]+[4]\Rightarrow 2=-6+5\mu$

$$\mu=\frac{8}{5}$$

$$2\lambda=2+\frac{24}{5}$$

$$\lambda=\frac{17}{5}$$

Substitute $\lambda=\dfrac{17}{5}$ and $\mu=\dfrac{8}{5}$ into [3] $\Rightarrow 1+\dfrac{51}{5}=-1-\dfrac{16}{5}$ which in inconsistent

Since the lines are not parallel and do not intersect, the lines are skew

4 $l_1:\dfrac{x-2}{3}=\dfrac{y-4}{2}=\dfrac{z-1}{2}$

$$\Rightarrow r=\begin{pmatrix}2\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\2\end{pmatrix},\lambda\in\mathbb{R}$$

$$r=\begin{pmatrix}0\\2\\3\end{pmatrix}+t\begin{pmatrix}-1\\4\\1\end{pmatrix}$$

$$\text{Equating}:\begin{pmatrix}2+3\lambda\\4+2\lambda\\1+2\lambda\end{pmatrix}=\begin{pmatrix}-t\\2+4t\\3+t\end{pmatrix}$$

$\Rightarrow 2+3\lambda=-t$ [1]

$4+2\lambda=2+4t$ [2]

$1+2\lambda=3+t$ [3]

Solving [1] and [3] simultaneously :

[1] + [3] $\Rightarrow$ 3 + 5λ = 3
$\lambda = 0$
Substitute into [1] $\Rightarrow$ 2 = –t
t = –2.
When $\lambda = 0$, t = –2, substitute into [2]
$\Rightarrow$ 4 = 4(–2) + 2
4 = –6
$\therefore$ l_1 does not intersect l_2
Since the lines are not parallel and do not intersect, the lines are skew

5 $x + 2y + z = 4$
$2x - y - z = 1$

The normal to the planes are $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\-1\end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix}1\\2\\1\end{pmatrix}.\begin{pmatrix}2\\-1\\-1\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\left|\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right|} = \frac{2-2-1}{\sqrt{6}\sqrt{6}} = \frac{-1}{6}$$

$\theta = 99.6°$

6 $\overrightarrow{OA} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$

$\therefore \left(\overrightarrow{OC} - \overrightarrow{OA}\right)(1-p) = p\left(\overrightarrow{OB} - \overrightarrow{OC}\right)$

$(1-p)\overrightarrow{OC} - (1-p)\left(\overrightarrow{OA}\right) = p\,\overrightarrow{OB} - p\,\overrightarrow{OC}$

$(1-p)\,\overrightarrow{OC} + p\,\overrightarrow{OC} = p(i - 2j + 3k) + (1-p)\,(2i + j - k)$

$\overrightarrow{OC} = pi + 2i - 2pi - 2pj + j - pj + 3pk - k + pk$

$= (2 - p)\,i + (1 - 3p)j + (4p - 1)k$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (i - 2j + 3k) - (2i + j - k)$

$= -i - 3j + 4k$

$\overrightarrow{OC}\,.\,\overrightarrow{AB} = 0$

$$\Rightarrow \begin{pmatrix} 2-p \\ 1-3p \\ 4p-1 \end{pmatrix} . \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix} = 0$$

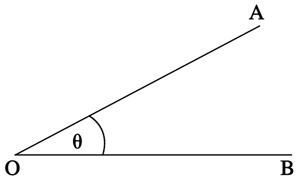
$\Rightarrow -2 + p - 3 + 9p + 16p - 4 = 0$

$26p = 9$

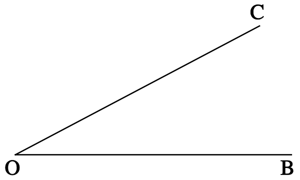
$$p = \frac{9}{26}$$

$$\therefore \overrightarrow{OC} = \left(2 - \frac{9}{26}\right)i + \left(1 - \frac{27}{26}\right)j + \left(\frac{18}{13} - 1\right)k$$

$$= \frac{43}{26}i - \frac{1}{26}j + \frac{5}{13}k$$



$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right|} = \frac{-3}{\sqrt{6}\sqrt{14}}$$



$$\cos\theta = \frac{\begin{pmatrix} 2-p \\ 1-3p \\ 4p-1 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}{\left|\begin{pmatrix} 2-p \\ 1-3p \\ 4p-1 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right|} = \frac{-3}{\sqrt{6}\sqrt{14}}$$

$$\frac{2 - p - 2 + 6p + 12p - 3}{\sqrt{(2-p)^2 + (1-3p)^2 + (4p-1)^2}\sqrt{14}} = \frac{-3}{\sqrt{16}\sqrt{14}}$$

$$\Rightarrow 17p - 3 = \frac{-3}{\sqrt{16}}\sqrt{(2-p)^2 + (1-3p)^2 + (4p-1)^2}$$

$$\Rightarrow (17p-3)^2 = \frac{3}{16}\left[(2-p)^2 + (1-3p)^2 + (4p-1)^2\right]$$

$$289p^2 - 102p + 9 = \frac{3}{16}(4 - 4p + p^2 + 1 - 6p + 9p^2 + 16p^2 - 8p + 1)$$

$4624p^2 - 1632p + 144 = 78p^2 - 54p + 18$

$4546p^2 - 1578p + 126 = 0$

$2273p^2 - 789p + 63 = 0$

$$p = \frac{789 \pm \sqrt{789^2 - 4(2273)(63)}}{2(2273)}$$

$$= \frac{789 \pm \sqrt{49725}}{4546}$$

$= 0.223,\ 0.125$

7 $\begin{pmatrix} 4+t \\ 9+6t \\ 12+5t \end{pmatrix} = \begin{pmatrix} -3+2s \\ -15+8s \\ -19+8s \end{pmatrix}$

$\Rightarrow 4 + t = -3 + 2s$ [1]

$9 + 6t = -15 + 8s$ [2]

$12 + 5t = -19 + 8s$ [3]

Solving [2] and [3] :

[2] – [3] $\Rightarrow -3 + t = 4$

$t = 7$

$\therefore\ 9 + 6(7) = -15 + 8s$

$s = 8.25$

Substituting into [1] $\Rightarrow 4 + 7 = -3 + 2(8.25)$

$11 = 13.5$

$\Rightarrow$ the lines do not intersect

8 $r = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 2+\lambda \\ -1+4\lambda \end{pmatrix}$

$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2+\lambda \\ -1+4\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ \lambda \\ 4\lambda \end{pmatrix}$

$\overrightarrow{AN} . \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -2 \\ \lambda \\ 4\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = 0$

$\Rightarrow \lambda + 16\lambda = 0$

$\lambda = 0$

$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$\overrightarrow{AN} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left| \overrightarrow{AN} \right| = \sqrt{(-2)^2} = 2$$

9 $l_1 : r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$$l_2 : r = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ -2\lambda \end{pmatrix} = \begin{pmatrix} 1-3\mu \\ 2 \\ 2+4\mu \end{pmatrix}$$

$\Rightarrow 1 + 2\lambda = 1 - 3\mu$ [1]
$-1 + \lambda = 2$ [2]
$-2\lambda = 2 + 4\mu$ [3]
From [2] $\lambda = 3$
Substitute into [3] $\Rightarrow -6 = 2 + 4\mu$
$\mu = -2$
Substitute $\lambda = 3$, $\mu = -2$ into [1] $\Rightarrow 1 + 6 = 1 + 6$
$\Rightarrow$ All three equations are satisfied by $\lambda = 3$, $\mu = -2$
$\therefore$ l_1 and l_2 intersect and hence l_1 and l_2 are not skew

10 $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$$\overrightarrow{OC} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$

(a) $\overrightarrow{OA} . \overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} . \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -2 + 3 + 1 = 2$

(b) $\cos\theta = \dfrac{\overrightarrow{OA} . \overrightarrow{OB}}{\left|\overrightarrow{OA}\right| \left|\overrightarrow{OB}\right|} = \dfrac{2}{\sqrt{11}\sqrt{6}} \Rightarrow \theta = 75.7°$

(c) $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-3\lambda \\ 3-2\lambda \\ 1 \end{pmatrix}$

$$\overrightarrow{OC} . \overrightarrow{AB} = \begin{pmatrix} 1-3\lambda \\ 3-2\lambda \\ 1 \end{pmatrix} . \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} = 0$$

$\Rightarrow -3 + 9\lambda - 6 + 4\lambda = 0$
$13\lambda = 9$

$$\lambda = \frac{9}{13}$$

(d) $$\left|\overrightarrow{OC}\right| = \sqrt{(1-3\lambda)^2 + (3-2\lambda)^2 + 1^2}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} \begin{pmatrix} 1-3\lambda \\ 3-2\lambda \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3\lambda \\ -2\lambda \\ 0 \end{pmatrix}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{9\lambda^2 + 4\lambda^2} = \sqrt{13\lambda^2}$$

$$\therefore \left|\overrightarrow{OC}\right| = \left|\overrightarrow{AC}\right|$$

$$\Rightarrow \sqrt{1 - 6\lambda + 9\lambda^2 + 9 - 12\lambda + 4\lambda^2 + 1} = \sqrt{13\lambda^2}$$

$$\Rightarrow 13\lambda^2 - 18\lambda + 11 = 13\lambda^2$$

$$18\lambda = 11$$

$$\lambda = \frac{11}{18}$$

11 $$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}$$

(a) $$\cos\theta = \frac{\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} . \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}}{\left|\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}\right| \left|\begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}\right|} = \frac{-21}{\sqrt{35}\sqrt{33}} \Rightarrow \theta = 128.2°$$

(b) Eq. of BC is

$$r = \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix} + \lambda\left(\begin{pmatrix} 5 \\ -2 \\ 11 \end{pmatrix} - \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ -6 \\ 12 \end{pmatrix}, \lambda \in \mathbb{R}$$

(c) $$r . \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} . \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$r . \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = -9$$

(d) $\cos\alpha = \dfrac{\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}9\\-6\\12\end{pmatrix}}{\left|\begin{pmatrix}-1\\1\\2\end{pmatrix}\right|\left|\begin{pmatrix}9\\-6\\12\end{pmatrix}\right|}\dfrac{9}{\sqrt{6}\sqrt{261}} \Rightarrow \alpha = 76.9°$

Angle between the line and the plane = 180 – (90 + 76.9) = 13.1°

12 $\begin{pmatrix}10\\t\\-t\end{pmatrix} = \begin{pmatrix}s\\5\\5s\end{pmatrix}$

$\therefore$ 10 = s

t = 5

–t = 5s

$\therefore$ –5 = 5(10) Inconsistent

Since the lines are not parallel and do not intersect they are skew

13 $\vec{OA} = \begin{pmatrix}3\\4\\-1\end{pmatrix}$, $\vec{OB} = \begin{pmatrix}5\\7\\6\end{pmatrix}$

(a) Direction of AB = $\begin{pmatrix}5\\7\\6\end{pmatrix} - \begin{pmatrix}3\\4\\-1\end{pmatrix} = \begin{pmatrix}2\\3\\7\end{pmatrix}$

Equation of the plane:

$r.\begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}3\\4\\-1\end{pmatrix}.\begin{pmatrix}2\\3\\7\end{pmatrix}$

$\Rightarrow r.\begin{pmatrix}2\\3\\7\end{pmatrix} = 11$

(b) $\vec{OA} = \begin{pmatrix}3\\4\\-1\end{pmatrix}$, $\vec{OC} = \begin{pmatrix}6\\23\\8\end{pmatrix}$

Equation of plane :

$r = \begin{pmatrix}3\\4\\-1\end{pmatrix} + \mu\left[\begin{pmatrix}6\\23\\8\end{pmatrix} - \begin{pmatrix}3\\4\\-1\end{pmatrix}\right]$

$r = \begin{pmatrix}3\\4\\-1\end{pmatrix} + \mu\begin{pmatrix}3\\19\\9\end{pmatrix}$

$\left.\begin{aligned}\therefore x &= 3 + 3\mu\\ y &= 4 + 19\mu\\ z &= -1 + 9\mu\end{aligned}\right\}\mu \in \mathbb{R}$

(c) Substitute the line into the plane :

$2(3 + 3\mu) + 3(4 + 19\mu) + 7(-1 + 9\mu) = 11$

$\Rightarrow 11 + 126\mu = 11$

$\mu = 0$

$\therefore$ The point of intersection is

$x = 3$

$y = 4$

$z = -1$

$$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

14 $\pi : 2x - 3y + z = 6$

$$l : r = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

(a) $$\left.\begin{matrix} x = 2 + t \\ y = 1 + 2t \\ z = -1 + 2t \end{matrix}\right\} t \in \mathbb{R}$$

(b) $2(2 + t) - 3(1 + 2t) + (-1 + 2t) = 6$

$\Rightarrow 2t - 6t + 2t = 6$

$-2t = 6$

$t = -3$

Point of intersection is $\begin{pmatrix} 2-3 \\ 1-6 \\ -1-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$

(c) $$\cos\theta = \frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right|} = \frac{-2}{\sqrt{9}\sqrt{14}} \Rightarrow \theta = 100.26°$$

Acute angle is $180 - 100.26 = 79.7°$

15 (a) $$\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$r = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} 3-3s \\ 4+s \\ 1+2s \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3-3s \\ 4+s \\ 1+2s \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-3s \\ 5+s \\ 2s \end{pmatrix}$$

Since AB is perpendicular to l:

$$\overrightarrow{AB} . \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1-3s \\ 5+s \\ 2s \end{pmatrix} . \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 9s + 5 + s + 4s = 0$$

$$14s = -2 \,,\, s = -\frac{1}{7}$$

$$\therefore \overrightarrow{OB} = \begin{pmatrix} 3+\frac{3}{7} \\ 4-\frac{1}{7} \\ 1-\frac{2}{7} \end{pmatrix} = \begin{pmatrix} 24/7 \\ 27/7 \\ 5/7 \end{pmatrix}$$

(b) $$\overrightarrow{AB} = \begin{pmatrix} 1+\frac{3}{7} \\ 5-\frac{1}{7} \\ \frac{-2}{7} \end{pmatrix} = \begin{pmatrix} 10/7 \\ 34/7 \\ -2/7 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(\frac{10}{7}\right)^2 + \left(\frac{34}{7}\right)^2 + \left(-\frac{2}{7}\right)^2} = 5.1$$